Innovaciencia
ISSN: 2346-075X Bucaramanga - Colombia

E- ISSN: 2346-075X
Innovaciencia 2023; 11(1); 1-19
http://dx.doi.org/10.15649/2346075X.3602
INVESTIGACIÓN CIENTÍFICA

Universal tools for analysing structures and interactions in geometry

Herramientas universales para analizar estructuras e interacciones en geometría.

Besim Berisha[1], Xhevdet Thaqi[2]*, Astrit Ramizi[3], Agon Mahmuti[4]

How to cite this paper: Berisha B., Thaqi X., Ramizi A., Mahmuti A., Universal tools for analysing structures and interactions in geometry. Innovaciencia 2023; 11(1): 1-19. DOI: 10.15649/2346075X.3602

Publicated: december 1, 2023

ABSTRACT

Introduction: This study examined symmetry and perspective in modern geometric transformations, treating them as functions that preserve specific properties while mapping one geometric figure to another. The purpose of this study was to investigate geometric transformations as a tool for analysis, to consider invariants as universal tools for studying geometry. **Materials and Methods:** The Erlangen ideas of F. I. Klein were used, which consider geometry as a theory of group invariants with respect to the transformation of the plane and space. **Results and Discussion:** Projective transformations and their extension to two-dimensional primitives were investigated. Two types of geometric correspondences, collinearity and correlation, and their properties were studied. The group of homotheties, including translations and parallel translations, and their role in the affine group were investigated. Homology with ideal line axes, such as stretching and centre stretching, was considered. Involutional homology and harmonic homology with the centre, axis, and homologous pairs of points were investigated. In this study unified geometry concepts, exploring how different geometric transformations relate and maintain properties across diverse geometric systems. **Conclusions:** It specifically examined Möbius transforms, including their matrix representation, trace, fixed points, and categorized them into identical transforms, nonlinear transforms, shifts, dilations, and inversions.

Keywords: geometric transformations; symmetry; perspective; invariance, Möbius transform.

INTRODUCTION

Symmetry and perspective are among the fundamental principles in scientific research, a tool for better understanding scientific theories and obtaining concrete results. Both leverage transformation mappings that preserve certain geometric properties. This invariance under groups of transformations allows classifying shapes, studying how structures change, and

[1] Faculty of Businesses, Haxhi Zeka University, 30000, UÇK Str., Peja, Kosovo. besim_berisha65@outlook.com
[2] Faculty of Applied Sciences, Public University "Kadri Zeka", 60000, Zija Shemsiu Str., Gjilan, Kosovo. xthaqi8@gmail.com
[3] Faculty of Applied Sciences, Public University "Kadri Zeka", 60000, Zija Shemsiu Str., Gjilan, Kosovo. astrit.ram1z1@hotmail.com
[4] Faculty of Applied Sciences, Public University "Kadri Zeka", 60000, Zija Shemsiu Str., Gjilan, Kosovo. amahmuti@outlook.com

revealing deep relationships between geometry and symmetry. Mathematical concepts like groups, isometries and projective geometry provide a unifying foundation. This paper discusses symmetry as a geometric regularity and a process of repetition. The first understanding of symmetry involves geometric patterns and relationships that reflect harmony and proportionality. This includes distinct types of symmetry, such as reflective, rotational, translational. The second understanding of symmetry is the idea of repeating elements, shapes, or images in a certain order. For instance, symmetry along the axis, where one part is a mirror image of the other.

Another productive tool of cognition is perspective. The development of the mathematical basis for the theory of perspective and two-dimensional representation of three-dimensional objects was initiated by the French mathematician J. Desargues, who in "Brouillon project d'une atteinte aux événements des rencontres d'un cône avec un plan" (1639) on conics described how objects are projected onto a plane with the help of a focus (the artist's imaginary eye) and what mathematical principles allow preserving the geometric structure of the original. The use of invariants provides a wide scope for developing algorithms and methods for automatic object recognition in images. For instance, in the field of computer vision, invariants help to separate objects from the background, determine their geometric characteristics and other important properties. Such studies are presented in many contemporary works.

Research by A. Barroso-Laguna et al.(1) in the field of pattern recognition and neural network training presents a new method Fundamental Scoring Network (FSNet) that evaluates the geometry of two views without relying on correspondences, which generates ideas for future research and improvements in the methodology and network design. The study of M. Buzzelli (2) deals with such a field of application of the analysed concept as data compression by reducing the number of colours in the palette and explores the use of invariants to small displacements in the colour space. This method can be useful in several areas where it is important to preserve image details with limited resources. However, more research and experimentation are needed to assess its performance and effectiveness on real data.

An interesting view is presented in the paper by C. Leopold (3), where the author proposes an innovative approach to the application of the mathematical concept of symmetry in the creative process. The study considers the concept of symmetry in terms of transformations and their properties and degree of invariance. The paper combines a mathematical understanding of symmetry as transformations with a more associative approach in the field of design. The concept is related to the theory of information aesthetics, where the interaction between redundancy and innovation becomes important for the aesthetic state, defined as the relation between the ordered and disordered state according to M. Bense (4).

The team of authors U. Gözütok et al. (5) developed and investigated a method for determining projective symmetries and invariants of rational curves by solving a system of polynomial equations. This method can be applied to curves of high degree and with large coefficients. The method can be extended to work with other geometric objects, such as surfaces, and higher dimensions, not limited to rational 3D curves. Further theoretical research in this area will offer a deeper insight into the mathematical foundations of this method and its connection with other geometric theories. G. Sergeant-Perthuis et al. (6) investigate how information about the spatial environment is perceived and processed by people. Their study focuses on theoretical aspects and concepts that can be used to understand the perception and behaviour of agents in virtual or robotic environments, which will help in the development of algorithms for teaching people and systems to use geometric structures to solve problems for space navigation, understanding spatial relations, and object analysis.

Thus, understanding the relationship between invariants, transformations, and properties of geometric objects contributes to the development of abstract and analytical thinking, which is necessary for solving complex problems both in scientific research and in various aspects of everyday life. The purpose of this study was to identify and analyse the invariants of geometric transformations, such as symmetry and perspective, and to use these invariants to classify different types of geometric shapes and highlight their common characteristics.

MATERIALS AND METHODS

Euclidean and Affine geometry

Research using the Erlangen ideas of F.I. Klein is an approach to geometric analysis that treats geometry as a theory of group invariants with respect to a transformation of the plane (or space). The approach helped to identify the properties of geometric objects that stay unchanged under certain group transformations, and thus change their symmetry and properties. The main components of the research methodology using Klein's Erlangen ideas include the following groups and methods.

A group is a mathematical structure consisting of a set of elements together with an operation ("·" or "*") that meets the requirements of associativity, closure, and the presence of identical and inverse elements.

Group transformations: a concept in geometry that is used to study the movements and changes in position of geometric objects in space. A transformation group is a set of all possible geometric transformations that can be applied to objects. The main components of group transformations are geometric objects – objects in geometric space (points, lines, shapes, polygons, circles) that can be viewed in two or more dimensions; and a group of transformations that includes all possible transformations that can be applied to geometric objects (shifts, rotations, reflections (symmetries), scaling); each transformation in the group is an element of this group. Rotation by angles around an axis forms a circular group. Reflections over mirror lines make up dihedral groups. Shears and dilations are other examples. Groups can be represented by abstract algebraic structures or tied to coordinate transformations through matrices, functions and equations.

Invariants: these are properties of geometric objects that stay unchanged after applying group transformations, i.e., they are characteristics that do not change during geometric operations. It provides formal machinery to analyse what structures are preserved under set actions. Properties invariant under groups reveal deep symmetries independent of coordinate choices. This enables a coordinate-free geometric approach focusing on essence rather than superficial aspects. Representation of groups: one of the basic concepts of group theory and mathematics, used to study groups, their properties and the interaction between its elements. Associativity – for any elements a, b, c of a group, the expression $(a - b)\ c$ is equal to $a\ (b \cdot c)$. A group has identical elements (denoted by e or 1), such that for any element a in the group, $e \cdot a = a \cdot e = a$. For each element a of the group, there is an inverse element a^{-1} such that $\mathrm{a} \cdot \mathrm{a}^{-1}$=$\mathrm{a}^{-1} \cdot$ a=e. Closed – the result of a transaction between any two elements of a group is also an element of that group. Thus, a group representation is a way of representing elements in the form of a matrix or other groups of objects, which is used for analysis and interaction with other transformations. This helped to investigate the properties of the group and its actions using algebraic and geometric methods.

Representation theory: methods of representation theory are used to analyse geometric transformations, which allows to reveal the structure of the group and find invariants. This theory uses mathematical concepts such as characteristics and eigenvalues to investigate the properties of group transformations. Using the invariants discovered during the analysis of group transformations, one can understand which properties of geometric objects stay unchanged and which can change. This made it possible to apply the symmetry of geometric systems and study their features. Using the method of F.I. Klein's Erlangen ideas, various geometries are classified based on their group invariants.

In conclusion, the application of F. Klein's Erlangen ideas to geometric analysis has been instrumental in deepening our understanding of geometry as a theory of group invariants with respect to transformations in space. This research methodology has provided a structured framework for investigating geometric objects and their properties in a way that transcends the specific geometric context. By emphasizing the concepts of group transformations, invariants, and representation theory, this approach has allowed us to discern the essential characteristics of geometric objects that remain unchanged under various transformations.

One of the contributions of this methodology is the ability to classify different geometries based on their group invariants. This classification not only helps us differentiate between simple transformations and collinearities but also provides insights into projective invariants, which have wide-ranging applications across various fields. The real-projective plane has emerged as a focal point in this research, offering a platform for exploring more general examples and extending the theory's applicability. Through the lens of Klein's Erlangen ideas, researchers have gained a clearer understanding of the symmetries inherent in geometric systems, enabling the study of their unique features and properties.

In summary, the adoption of F.I. Klein's Erlangen ideas in geometric analysis has provided a powerful framework for investigating the fundamental principles that govern geometric objects' transformations and invariants. This approach continues to be a subject worthy of attention, offering valuable insights and applications in diverse fields of mathematics and beyond.

RESULTS

The connection between symmetries and structures is that the symmetries of an object preserve the properties and operations that are defined within a given structure. This is a key concept in mathematics that allows analysing and understanding distinct types of transformations and their impact on mathematical objects. Transformations can be symmetric (preserve structures (rotations and transfers in the plane)), and not symmetric (homotheties that can change the scale of objects) [7]. Examples such as rotations, reflections and shifts show how the structure of a group generalises the idea of symmetry [8].

For the case when S is a space, the following examples are considered. Diverse types of mathematical objects, such as vector or topological spaces, have defined structures and operations. The symmetries of these objects are transformations that preserve these structures and properties (1):

$$\Phi(x+y) = \Phi(\mathrm{x}) + \Phi(\mathrm{y}), \Phi(\lambda \mathrm{x}) = \lambda\Phi(\mathrm{x}) \tag{1}$$

If S is a vector space, then the S symmetries are isomorphisms, these transformations preserve the linear structure of the vector space. This implies that the operations of vector addition and scalar multiplication are preserved after applying symmetries. In the case of a topological structure, the S symmetries are represented by homeomorphisms that preserve topological properties or by continuous reversible transformations whose inverse is also continuous. An abstract group is considered as a group of transformations of spaces M, using the concept of group action on a set. Each element g from the abstract group G is assigned a transformation Φg acting on the elements of the space M. This action can be specified by certain rules, depending on the context of the task. The neutral element e of an abstract group G must correspond to the identity transformation of M, i.e. (2):

$$\Phi_{\mathrm{g}} = \mathrm{idM} \tag{2}$$

The product of elements g_1 and g_2 in the abstract group G corresponds to the composition of the transformations $\Phi g1$ and $\Phi g2$ in the space M, i.e. (3):

$$\Phi_{g1} \cdot \Phi_{g2} = \Phi_{g1*g2} \tag{3}$$

If M is structured, one can obtain realisations of G in which Φg is symmetric for every $g \in G$. Thus, the realisation of an abstract group as a group of transformations of a space M allows investigating the structure of the group through its action on the elements of the space. This concept plays a vital role in algebra, geometry, and other areas of mathematics where symmetries and transformations are studied. Perspective is another fundamental element of projective geometry. Projective transformations used in perspective allow three-dimensional objects to be depicted on a plane in a way that preserves distance relationships and linear structures. These include such transformations as projective collinearities and projective correlations, which investigate the properties and relationships between points and lines in space after applying projections, where the concepts "line" and "parallelism" may look different than in conventional Euclidean geometry.

New research in the field of invariants associated with geometric configurations consists of theoretical studies and practical applications for solving computer vision problems. The first area involves the study of properties of invariants: improving theoretical approaches to understanding invariants, specifically their stability (response to small changes) and numerical stability (sensitivity to computational errors). Computing new invariants are developing new methods for finding invariants that apply to different geometric configurations, such as curves, lines and conics, possibly in three dimensions(9, 10). The definition of an invariant is given: it is a numerical value defined for a geometric configuration α that stays unchanged when a geometric function Ψ is applied to this configuration, i.e., $f(\alpha)$ stays unchanged after the application of the function Ψ (4):

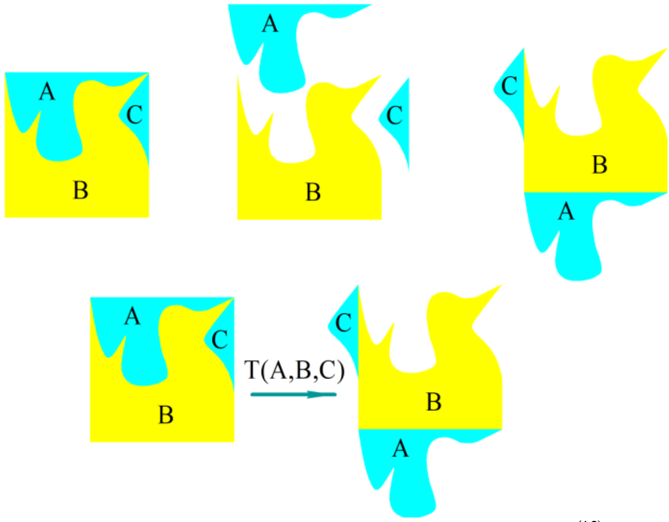
$$\mathrm{f(a)} = \mathrm{f}(\Psi(\mathrm{a})). \tag{4}$$

If a second configuration β exists, and $f(\beta) = f(\alpha)$, then the configurations can be potentially equivalent. If the configuration contains several subconfigurations $\alpha = (\alpha 1, \alpha 2, \dots, \alpha n)$, the invariant can be a prediction function. If the subconfigurations $\alpha 1, \alpha 2, \dots, \alpha n\text{−}1$ and $f(\alpha)$ are known, then the value αn can be calculated, for this the function f must be inverted. The nature of the invariants existing in a particular problem depends on the geometric set of functions. This is the case for diverse types of invariants, such as angles and length ratios associated with similarity; length ratios of collinear points play the same role in affine transformations as the cross ratio of parallel lines or collinear points in the case of collinearities. The main task is to find invariant functions that exist in particular problems. The order in which the functions are applied does not matter for properties. If a function $g \circ f$ stays invariant for any real function g, it means that the invariance is preserved regardless of what particular functions are applied to f.

Keeping in mind the Gross-Kuan principle that each element of a set G is a function that takes input from the configuration space E and produces a result also in the space E, a particular configuration x belonging to E was considered, and various transformations from the set G were applied to it(11). Next, the following questions are investigated: is it possible to obtain all possible configurations of the space E, proceeding from a single configuration x? If not, is there a description of the set of configurations that can be obtained from x? And in this case, how many independent parameters are required for this?

These independent parameters are invariants for the set of transformations G, which can be described as functions that map the elements of the set of transformations G to the set of real numbers. Several hypotheses are put forward for the analytical calculation of the invariants of a given case. If a set G is a group acting on E, and the elements of G are functions from E into itself, then G can be described by the function composition operation, which is denoted either by the symbol "∘" or multiplicatively. The description also includes axioms that can be used to prove the group. Then the concept of a group acting on a set is as follows. A group acting on a set consists of functions from the set to itself and meets the properties of the group. For instance, rotations around a point form a group that acts on a plane. It is advisable to use real parameters to analyse the invariants by computing in the configuration set. Examples of invariants include angles for plane rotations, parameters for plane displacements, and axis-related parameters for 3D displacements(10). Figure 1 illustrates an example of an equivalent transformation of one flat shape into another while preserving the surface area, with the surface area being invariant.

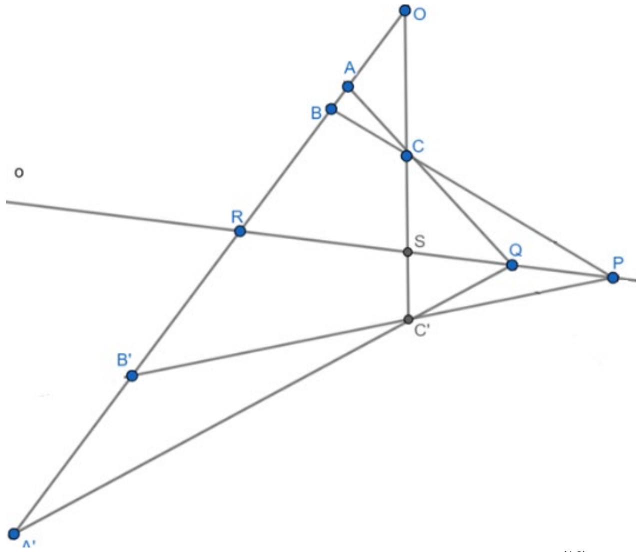
Figure 1. An example of an equivalent square to shape conversion



Source: compiled by the author based on (12).

Architecture of geometry through geometric transformations. The space X contains points as elements and its subsets as shapes. Transformations are bijective mappings f: X→X. Next, projective transformations are discussed, in which all other forms are created from one-dimensional forms. This extension will help to systematically link projective and Euclidean geometry. Euclidean geometry is the study of flat shapes or figures of flat surfaces and straight lines in two dimensions. The study considered the extensions of projectivity to two-dimensional primitives such as point fields, line fields, plane bundles and linear bundles, with a focus on point and line fields in two-dimensional space and planar perspective. Planar perspective is a geometric transformation where points are converted to lines and vice versa, with the important concepts of perspective axis and perspective centre. The concept of planar perspective has diverse variations, including planar homology (axis and centre do not coincide), planar elation (axis and centre coincide), and involutional or harmonic homology (homologous pairs of points map onto each other in pairs).
The centre 0, the o-axis and the homologous points A and A' were considered. A homologue B was constructed for any point B of the plane (Figure 2). The first case is when the point B does not coincide with the line OA (or OA'). If line AB intersects the o axis at point R, then R is a fixed point. Thus, the line A'B' must also intersect the o axis at the point R. Homologue B' is defined as the intersection of line OB and line A'R and is therefore unique. It is also possible to construct a pair of homologous points C and C' if point B is on the line OA, but using the pair C, C' instead of A, A'. Then the homologue B' will be constructed in an analogous way, i.e., it is proved that the construction of the point B' does not depend on a certain pair C, C', chosen initially. The same reasoning applies to the case of planar homology.

Figure 2. Plotting homologous points in a plane perspective



Source: compiled by the author based on [13].

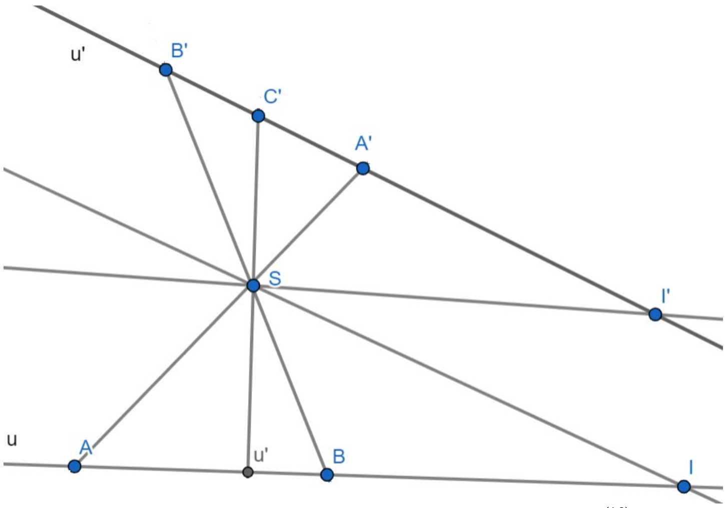
Collinearity and correlation are two diverse types of (1,1)-matching in geometry, and they differ in the nature of the homologous elements and the way they interact as follows. In collinearity, homologous elements belong to the same type of geometric objects (a point is homologous to a point, a line is homologous to a line). For every one-dimensional primitive, there is a corresponding one-dimensional primitive – every point has its homologous point, and every line has its homologous line. In correlation, homologous elements can establish various correspondences between points, lines, and other objects. One of the exceptional cases of correlation is the relationship between points and their polar lines on a given conic curve in the plane. For instance, the set of transformations F, G, H has the following properties: the product of any two transformations from this set and the inverse of each transformation are contained in the set itself. To transform F, it is important that the F^{-1} reversal operation is identical, i.e., does not change the result. It is also noted that the operation in this set is associative, which means that the order of operations does not affect the final result (5):

$$\mathrm{F(GH) = (FG)H = FGH.} \tag{5}$$

This set of operations, as described, creates a group of operations. For instance, projective transformations preserve the relative position of points and have the properties of composition and inverse transformation. Collinearities (points on the same line) and correlations (combinations of collinearities) form a group, but flat correlations are not a group because of the presence of inverse elements. The general projective group of area transformations includes all collinearities and correlations, creating an important mathematical structure. Affine geometry is a mathematical system that results from Euclidean geometry by ignoring the metric notions of distance and angle. It is a geometry in which properties are preserved by parallel projection from one plane to another, and it focuses on the relationships between points, lines, and higher-dimensional subspaces without considering their lengths or angles. Affine geometry studies the properties of shapes that stay unchanged under a certain group of transformations called the affine group. This group consists of projective collinearities that preserve the selected line on the plane, known as the invariant line. The points on this line are not necessarily invariant, and these properties differ from those preserved by a more general group of transformations known as collinearities.

Projective geometry deals with perspectives and projections, treating parallel lines equivalently. Affine geometry presupposes an invariant line at infinity, enabling parallel line and vector concepts resembling Euclid's axioms. This perfect line provides a reference differentiating affine maps from general projective transformations. Ordinary points and lines make up the Euclidean plane, while the perfect line and its points play a key role in affine geometry. This perspective shows the inclusion of the Euclidean plane in a wider projective plane.
The theorems and properties of affine transformations state as follows: firstly, there is a single affine transformation that transforms the ideal points A, B, C into the ideal points A', B', C'; at follows that parallel lines are transformed into parallel lines under the action of an affine transformation. Secondly, the ratio of division of segment AB by point C stays unchanged under the affine transformation. Tthirdly, the ratio of division of segment AB by point C stays unchanged under the affine transformation. Figure 3 illustrates this: the line l with points A, B, C on it is transformed into a line l' containing A', B', C'.

Figure 3. An invariant with respect to the affine transformation.



Source: compiled by the author based on [13].

In addition, I, I' are respectively the intersections of l, l' with the ideal line, and (6):

$$(\mathrm{AB}, \mathrm{CI}) = (\mathrm{A'B'}, \mathrm{C'I'}), \tag{6}$$

i.e. (7):

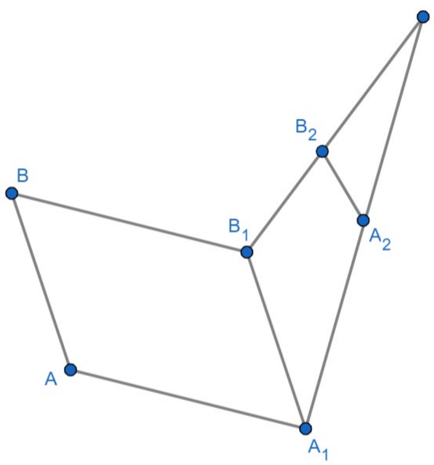
$$\frac{\mathrm{AC}}{\mathrm{CB}} = \frac{\mathrm{A'C'}}{\mathrm{C'B'}} = \mathrm{const.} \tag{7}$$

This research methodology has led to the identification of properties of geometric objects that remain unchanged under specific group transformations, shedding light on the fundamental principles governing symmetry and structure in geometry. Key components of this methodology, including group transformations, invariants, and representation theory, have been instrumental in classifying various geometries based on their group invariants and exploring their wide-ranging applications. This approach has facilitated the study of symmetries, transformations, and invariants in different mathematical contexts, from projective and affine geometry to realizations of abstract groups, resulting in valuable insights and practical applications in fields like computer vision and mathematical modelling. Moreover, the exploration of collinearity, correlation, and their connections within this framework has enriched our understanding of geometric transformations and their impact on mathematical objects, contributing to the ongoing advancement of geometric analysis.

The Möbius transformation

The geometry of homotheties. In a homothetic group, translations occur along an ideal straight axis, while parallel translations occur on ordinary lines passing through its centre. This group forms a subgroup of the affinity group containing unique translations defined by centres, axes, and pairs of homologous points. Transmissions keep the lines passing through the centre unchanged but shift the other lines into parallel ones. Homologies with perfect line axes are either extensions, or extensions centred around 0. Sprains and shears that are performed together do not always constitute a group. This union forms a homothetic group of flat transformations, a subgroup of affine transformations. The geometric properties that are preserved under homothetic transformations are called homothetic properties, which defines a homothetic geometry where similar, equally spaced shapes predominate. The homothety group covers transformations that are combined in such a way that the result is an extension or translation (Figure 4).

Figure 4. Translations and stretching



Source: compiled by the author.

If T is a translation centred on O_1, and S is a stretching centred on O_2, then the following is obtained (8-10):

$$\mathrm{T(A) = A_1, S(A) = A_2,} \tag{8}$$

$$\mathrm{T(B) = B_1, S(B_1) = B_2,} \tag{9}$$

$$\mathrm{T(AB) = A_1B_1, S(A_1B_1) = A_2B_2 \text{ i } ST(AB) = A_2B_2.} \tag{10}$$

Thus, if ABB_1A_1 is a parallelogram, and the segments AB and A_1B_2 are equal, then when S is stretched, there is $A1B1 \| A2B2$, while $\frac{|\mathrm{A1B1}|}{|\mathrm{A2B2}|} = \mathrm{const}\ \mu$ for the S transformation. The stretching coefficient of the transformation $\mu = const$ indicates how much the segment is stretched or compressed under the action of the homothetic transformation. Thus, the theorem states that under a homothetic transformation, the segments that correspond to each other after the transformation – homologous segments – stay parallel. Moreover, the ratio of the lengths of these homologous segments is constant and equal to the stretching

factor μ. This theorem establishes a connection between homothetic transformations, stretching, and the properties of segments in geometry.
Geometry of similarity. The perfect line does not change under the affine transformation, but it does not leave the points on the line unchanged. The homothetic transformation, on the other hand, preserves the perfect line along with all the perfect points. Next, the study considered a subgroup of the affine group that includes an elliptic involution on an ideal line. This involution, known as absolute or orthogonal involution I, transforms the lines containing the corresponding ideal points into perpendicular lines [14]. This involution can be any elliptical. Defining an orthogonal involution, one can say that the group of projective collinearities that keep it invariant is a similarity group. The properties preserved by this group, rather than the wider affine group, are called similarity properties, which define the geometry of similarity. This type of geometry is part of the broader Euclidean geometry, which includes both similar and congruent figures. Congruent figures are geometric figures that have the same shape and size, meaning they can be superimposed exactly on top of each other. As a result, the homologous points under I remain so. This leads to the theorem that perpendicular and parallel lines remain so under the similarity transformation. This means that circles are also transformed into circles by similarity.
Metric geometry. Next, the study considered involutional homology. Another name for it is harmonic homology – it has the centre 0, the o axis and different points A, A' as homologues [14]. The pair A, A' is a double correspondence, and therefore A' is transformed into A. Thus, if $A' \cap o = B$, then (11, 12):

$$\mathrm{AA'OR} \underset{\wedge}{-} \mathrm{A'AOR}, \tag{11}$$

$$(\mathrm{AA'}, 0\mathrm{R}) = (\mathrm{A'A}, 0\mathrm{R}), \tag{12}$$

furthermore (13, 14):

$$(\mathrm{AA'}, 0\mathrm{R}) = \frac{1}{(\mathrm{A'A}, 0\mathrm{R})}, \tag{13}$$

$$(\mathrm{AA'}, 0\mathrm{R}) = \pm 1. \tag{14}$$

The properties of involutional or harmonic homologies are discussed with an emphasis on reflections of orthogonal lines. In the case of involutional or harmonic homology, the homologous points are arranged in such a way that they form a special distribution, in which the centre of homology and the point of intersection of the given lines with the main axis play a significant role. This configuration leads to the possibility of determining the reflection of an orthogonal line in the form of a harmonic homology with certain characteristics. In an orthogonal line mapping, homologous segments have the same length, and homologous angles have equal values but opposite signs. These transformations preserve the distance between points, as well as the magnitude and sign of angles. A series of orthogonal linear mappings results in displacement or rigid motion, forming a metric plane group. For a group of rotations around a fixed point, any rigid motion is a combination of rotation and translation. Thus, the metric group is a special case of the projective group, which is restricted by additional conditions on the preservation of distances and angles. Every operation of the metric group is also a projective transformation, but not vice versa. Möbius transformation. To define a group of transformations from a set of hyperbolic transformations H that map hyperbolic directions to other hyperbolic directions, two seemingly different types of directions are combined: there are directions represented both in Euclidean space and on Euclidean circles [15].
The study considered the stereographic projection for the points of the unit circle S_1, except for the point i (complex unit). The essence of the projection is that each point z on the unit

circle corresponds to a point on the real axis R. This correspondence is defined by the line connecting point i and point z. The function ξ carries out this correspondence and is called a stereographic projection. Notably, this function can be exactly defined for all points except point i, and the real part of point z (Im(z)) is not equal to 1 (Figure 5). If there are two points on the plane (which are, in fact, straight lines), one can construct their corresponding Euclidean circles by adding one more point to these points, which will be the centre of each circle. Thus, Euclidean circles can be represented as a combination of Euclidean lines and added points that define the centres of the rings. The inclusion of this additional point in the complex plane C leads to what is called the Riemann sphere, which in complex analysis is denoted as C_r. It is a tool that allows considering complex functions on the C plane, including functions that have poles or isolated special points.

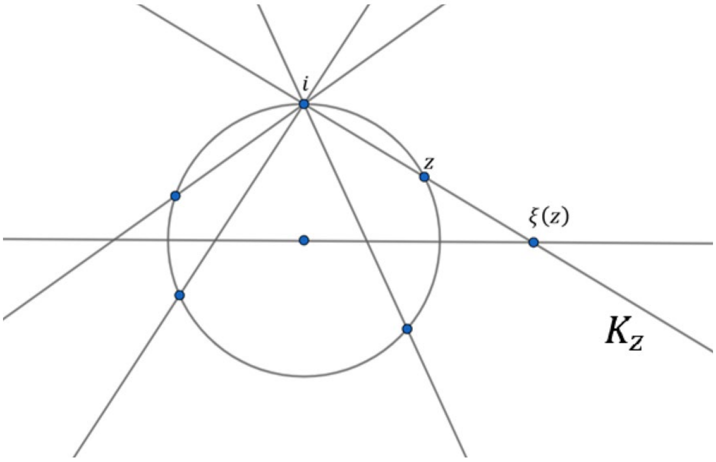


Figure 5. Stereographic projection
Source: compiled by the author.

The Riemann sphere C_r is the union of the complex plane C and the point at infinity (∞). A subset X is considered open in C if it includes all points, while a set X is classified as closed if its complement lies outside C[16]. Definition: Let a, b, c, d be complex numbers and satisfy the condition (15):

$$\mathrm{ad} - \mathrm{be} \neq 0. \tag{15}$$

The Möbius transform is a function $T: C \to C$ (16):

$$\mathrm{T(z)} = \begin{cases} \frac{\mathrm{az+b}}{\mathrm{cz+d}};\ \text{where } \mathrm{z} \neq -\frac{\mathrm{d}}{\mathrm{c}}, \infty \\ \infty;\ \text{where } \mathrm{z} = -\frac{\mathrm{d}}{\mathrm{c}} \\ \frac{\mathrm{a}}{\mathrm{c}};\ \text{where } \mathrm{z} = \infty \end{cases} \tag{16}$$

The variable z is part of a subset X, and for it there exists a positive integer ε such that the entire open disc (17):

$$\mathrm{U}\varepsilon(\mathrm{z}) = \{\mathrm{w}\ \in \mathrm{C}\ \|\mathrm{w} - \mathrm{z}| < \varepsilon\}, \tag{17}$$

also belongs to the set X.

In the following, separate types of Möbius transformations, which are parts of $SL_2(C)$ groups (Special Linear Group over complex numbers), are considered. Möbius transformations, defined as (18):

$$T(z) = (az + b)/(cz + d), \tag{18}$$

for certain values of a, b, c, and d include distinct types of transformations on the complex plane.

Identical transformation: if a and d are equal to 1 in a Mobius transformation, and b and c are equal to 0, then this transformation becomes identical and does not change the value of the variable z. In this special case, the Möbius transform simply reduces to a function that leaves z unchanged. Möbius nonlinear transformations: if c=0, then (19):

$$\mathrm{T(z) = a/d + b/d}, \tag{19}$$

is a nonlinear Möbius transformation that cannot be expressed in the form $Az + B$. These transformations constitute a group.
Shift: another special case of the Möbius transformation is that if a and d are equal to 1 and c is equal to 0 in the Möbius transformation, then this transformation becomes an operation of shifting the variable z by b units to the right on the complex plane.
Expansion: also, a special case of the Möbius transform – if in the Möbius transform the values b and c are 0 and d is 1, then this transform becomes an expansion (or scaling) operation by a, where a is a constant that changes the scale of the variable z on the complex plane. That is, all points on the plane will be stretched or compressed relative to the point (0.0) by a distance a.
Inversion: another special case of the Möbius transformation – if in the Möbius transformation a and d are 0, and b and c are 1, then this transformation becomes an inversion operation. The inversion maps a point z to its inverse $1/z$, i.e., each point on the complex plane will be mapped to its inverse value relative to the origin. Properties for a matrix that is a representation of the Möbius transform in matrix form (20):

$$\mathrm{AT} = \begin{pmatrix} \mathrm{a\,b} \\ \mathrm{c\,d} \end{pmatrix}. \tag{20}$$

The inverse of the Möbius transformation corresponds to the inverse of the matrix $\mathrm{A_T^{-1}}$ and has a corresponding associated Möbius transformation T^{-1} A Möbius transformation $T(z)$ is bijective, i.e., mutually unambiguous, if and only if det $AT \neq 0$. The composition of the Möbius transform corresponds to the multiplication of their respective matrices (21):

$$\mathrm{A_T \cdot B_T = AB_T}. \tag{21}$$

In conclusion: the set of Möbius transformations $T(z)$ is a homomorphism of the group $SL_2(C)$. Definition of the Möbius Transform Matrix – this matrix represents the Möbius transform in matrix form. The shape of this matrix is as follows (22):

$$\mathrm{M = (ab; cd)}, \tag{22}$$

where: a, b, c and d are complex numbers such that the determinant of the matrix M is equal to 1 and the matrix belongs to the group $SL_2(C)$, which consists of 2x2 complex matrices with a determinant of 1.
Next, some important aspects of the Möbius transform were considered, specifically, its trace, fixed points, and their classification. Firstly, to calculate the trace of the Möbius transform, one needs to sum the four coefficients of the matrix located on its main diagonal (i.e., a, d, b and c). Secondly, fixed points can be calculated using a trace (T_rT). Fixed points are points for which the transformation value is equal to their eigenvalue: $T(z) = z$. Thus,

when using the Möbius trace, the formula for finding such points takes the following form (23):

$$Z_{1,2} = \frac{a-d \pm \sqrt{(d-a)^2+4bc}}{2c}. \tag{23}$$

To calculate the expression under the root, the property $ad - bc = 1$ is used, and it turns out that it is equal to $(a + d)^2$-4, which also coincides with the trace of the transformation. Next, the practical use of the trace of the Möbius transform (T_rT) for finding fixed points and establishing criteria for classifying these transforms based on their characteristics was considered. Loxodromic transformations are Möbius transformations with 2 fixed points. They have certain characteristics: $Tr(T)$ must belong to the set of complex numbers, excluding the interval [-2, 2]. In addition, such transformations are complexly conjugate to the expression $T(z) = kz$, where $|k|>1$. The corresponding transformations on the complex plane look like a double helix (Figure 6).

Figure 6. Double helix

Source: compiled by the author.

Hyperbolic transformations are also Möbius transformations with 2 fixed points. The trace of the transformation is the real number $Tr(T)$, which belongs to the set of complex numbers, except for the interval [-2, 2]. Such transformations are conjugate to the expansion $T(z) = kz$, where k is a real number and $k > 1$. In other words, hyperbolic transformations can be considered as a special case of loxodromic transformations, where points move in a circle, passing one fixed point and reaching another. Elliptic transformations are also Möbius transformations with 2 fixed points. Their characteristics are that the trace of the transformation $Tr(T)$ is a real number from the interval [-2, 2], and they are conjugate to the rotation $T(z) = kz$, where $|k| = 1$. The elliptical transformation looks like an elliptical movement of points around two unchanged foci. Parabolic transformations are Möbius transformations with one fixed point. They are characterised by the requirement $Tr(T) = \pm 2$ and their conjugation is performed by means of a shift $T(z) = z + a$. During the parabolic transformation, there is only one fixed point, and the orbits of the points come from it and return to it along a circular trajectory. The coefficient k, the value of which changes the shape and size of the object to which the transformation is applied, is also called the Möbius transformation multiplier. Exceptional cases are the Möbius transformation with more than two fixed points and the trivial Möbius transformation. The latter is an identical reflection when each point stays in place. The fixed points of an identical transformation are all its points.

In conclusion, the exploration of homotheties and their role in geometric transformations has provided valuable insights into the geometry of similarity and metric geometry. Homothetic transformations, characterized by the preservation of parallelism and constant stretching factors, form the basis for understanding homothetic properties and the geometry of similarity, where similar shapes with equal spacing predominate. These transformations play a fundamental role in connecting the properties of segments in geometry to stretching factors. Additionally, the study delved into orthogonal involutions and their significance in defining similarity properties, highlighting the preservation of perpendicular and parallel lines, as well as circles, under similarity transformations. Moreover, the investigation of harmonic homologies revealed their distinctive characteristics, including the preservation of distance and angles, leading to the formation of metric plane groups. The Möbius transformations further enriched the geometric framework by introducing hyperbolic, elliptic, and parabolic transformations with fixed points and specific trace characteristics. The understanding of these diverse transformations and their relationships contributes to a deeper comprehension of geometric structures and their preservation properties, offering valuable insights into various mathematical and geometric contexts.

DISCUSSION

Understanding the relationship between invariants, transformations, and properties of geometric objects is fundamental to developing deep and flexible mathematical thinking, as well as to applying geometry in various sciences and engineering. Invariants help to identify and analyse the essential aspects of objects that stay constant regardless of their location or orientation. This makes it possible to classify and compare different objects. Transformations – reflections, rotations, shifts – help study how objects change under the influence of various operations and how these changes affect their properties. The relationship between invariants, transformations, and properties allows revealing the deep relationships between all aspects of geometric objects, making it possible to measure and compare objects regardless of their location. In computer graphics and image processing, the concepts of invariants and transformations are used to develop algorithms for pattern recognition, data compression, and in simulations and visualisation.

J. Flusser et al. (17) investigate the issue of image degradation (blurring) and proposes an approach to restoring such images by solving the defocusing problem and using deep learning through data augmentation. An alternative way is proposed to describe and recognise blurred images without using defocusing and expanding datasets. Using the theory of invariants with respect to blurring, it is proposed to construct characteristics (invariants) of images in Fourier space, using orthogonal projection operators and moment expansion for efficient computation. It is also pointed out that this new theory of invariants does not require prior knowledge of the type of blur. By applying the general rule of substitution, invariants combined with invariants relative to spatial transformations are easy to construct and use. The study is interesting for the experimental comparison of the proposed theory with conventional Convolutional Neural Networks, which demonstrates the advantages of the proposed theory. It is considered a controversial point to discuss the advantages and limitations of the proposed theory in comparison with other methods: what are the advantages of this theory in blurred image restoration, and why can it be more effective or less effective in certain conditions?

The paper by D. Khadjiev and İ. Ören (18) is a study in geometry and studies the global G-invariants of plane trajectories and plane curves in the two-dimensional Euclidean space E_2 for the orthogonal group $G=O(2)$ and the special orthogonal group $G=O^+(2)$. It shows how objects change under the influence of these transformations. The use of complex numbers to understand the equivalence between broad trajectories under the action of this group of transformations is discussed. It can be agreed that the results of this study have further prospects in both fundamental research and applied development, although it is limited to two-dimensional space and the results may not be applicable to higher dimensions. It is also worth noting the high computing power requirements for calculating global invariants and transformations between objects [19].

The study by C. Kanbak et al. [20] investigates the invariant properties of deep neural networks, specifically their resistance to geometric data transformations. The ManiFool algorithm proposed by the authors is a way to measure these invariant properties and is used to analyse and improve the invariance of neural network models. The authors' research also provides another important component related to adversarial learning, which improves the invariant properties of deep neural networks. Admittedly, this study has prospects for research in the field of invariance of neural networks under geometric transformations, and contributes to the development of computer vision, machine learning, and image processing security by helping to understand and improve the invariance of deep networks to geometric transformations. However, it is important to realise that ManiFool may have limitations in what types of geometric transformations it can study and measure. This may affect the application of the method in particular situations and require additional tools to analyse invariance to a wider range of transformations.

G. Berton et al. [21] describe the GeoWarp method, which is designed for visible geolocation. One of the key features of this method is that it considers shifts in viewpoints when comparing objects in images. This means that the method makes it possible to consider the projective transformations that are revealed by changes in viewpoints when determining a location based on visual data. It is noted that this approach has the advantage of being resistant to changes in illumination and the closure of parts of objects, but does not consider the shift in the viewpoint, which is the number of aspects in geolocation [22; 23]. An interesting continuation of this study could be a comparison with other methods, a forecast of effectiveness in practice, and an understanding of possible limitations.

An interesting and informative review is the paper by E. Bayro-Corrochano [24], which discusses geometric algebra as an advanced mathematical tool. The text is a survey of papers on quaternionic algebra and applications of geometric algebra in computer science and engineering from 1995 to 2020. The author discusses various areas of application of geometric algebra. The characteristics of its use for solving problems in computer science and engineering are analysed. The challenges and prospects of GA application proposed by various researchers are considered. Particular emphasis is placed on the development of geometric algebra in the fields of image processing, computer vision, neurocomputing, quantum computing, robot modelling, control and tracking, and improving the performance of computing [25; 26]. This detailed review will contribute to the further development of geometric computing for intelligent systems.

The final vision of the applied use of research in computer vision and projective geometry includes the development and improvement of algorithms for recognising objects in images, ensuring that their geometric properties and symmetry are reproduced with accuracy [27; 28]. This is achieved by analysing and identifying invariants during transformations of objects in the space of view changes. One can agree that the key aspects of this research are the study of perspective and projective geometry for accurately mapping three-dimensional objects to two-dimensional images from different viewpoints. The paper by R. Velich and R. Kimmel [29] proposes an innovative approach to the numerical approximation of differential invariants of plane curves. The study uses the universal properties of deep neural networks to estimate geometric quantities. The authors provide other researchers in the field with a set of test data on which to further test and evaluate similar models. In this regard, the study can be considered promising, as such a dataset can be useful for comparing different approaches and determining their accuracy and efficiency.

C. Zhao et al. [30] focused on the analysis of objects represented as a point cloud in three-dimensional computer vision. The main objective of this study is to develop a method for creating a representation that remains unchanged when objects are rotated in three-dimensional space. The researchers propose an approach to achieve rotational invariance (RI) by combining local geometry and global topology. Their local-global-representation network (LGR-Net) has two branches: one encodes local geometric RI features, while the other encodes global topology-preserving RI features. The authors emphasise that local geometry and global topology have different but complementary properties with respect to RI in different regions of three-dimensional space. The presented experimental results show that LGR-Net achieves the best performance on different datasets containing objects with different rotations. The study makes a major step towards the development of methods for analysing point clouds that consider rotational invariance, which is an urgent problem in the field of three-dimensional computer vision.

In the study by D. Rudrauf et al. [31] the projective geometry apparatus is used to explain the Moon Illusion and create a model of consciousness. At first glance, this paper may appear rather abstract and theoretical. However, these results may be further developed in psychology, neuroscience, and cognitive sciences in the study of consciousness and perception. This may also have practical applications in the development of optical systems, as the study emphasises that projective geometry is useful for explaining optical illusions [32; 33]. The team of authors E. Gorda et al. [34] in investigates approaches to modelling digital images, specifically in the field of localisation and object identification using data reduction. The paper focuses on the topology of a discrete two-dimensional image within the framework of the problem of determining the invariants of diffeomorphic transformations. The results can be used to automate and improve the processing of large amounts of visual data in various fields: security, autonomous vehicles, medical image analysis.

Further research is needed to assess the rotational invariance method's effectiveness for 3D point clouds in several key areas [35; 36]. First, more comprehensive benchmark datasets should be developed to evaluate the method's performance across various real-world scenarios and object types. Additionally, studies should investigate the scalability of the approach to handle larger and denser point clouds efficiently. Moreover, exploring techniques to adapt the method for different sensor modalities (e.g., LiDAR, depth cameras) and addressing robustness to noise and occlusion in 3D data are crucial research directions [37]. Lastly, investigations into the method's generalization capabilities across different application domains, such as robotics, computer vision, and augmented reality, would help establish its broader utility.

To summarise, research in computer vision and projective geometry with a focus on design, invariants, symmetry, and perspective opens a wide range of possibilities in developing advanced algorithms for object recognition, image analysis, rendering three-dimensional scenes in two-dimensional formats and improving the visual experience in various technology areas.

CONCLUSIONS

The investigation into various approaches to the classification of geometries and the crucial role of invariants in delineating geometric properties across diverse transformation types unveils a multifaceted landscape in mathematical analysis. The classical Euclidean geometry, rooted in axioms and deductive reasoning, served as the foundational paradigm for centuries. Nevertheless, as mathematics evolved, emphasizing numerical and set-theoretic aspects, the classical approach demonstrated limitations in addressing contemporary geometric challenges. The analytical approach to geometry, incorporating algebraic and analytical techniques, offers a potent alternative. It enables the simultaneous examination of multiple geometric spaces, fostering the exploration of nuanced properties such as metric and curvature characteristics. Central to this exploration is the concept of isometries, which underpin the preservation of fundamental spatial attributes, irrespective of location or orientation. Isometries emerge as a pivotal tool in unveiling invariant geometric properties, especially across distinct geometric structures.

Projective geometry emerges as a dynamic framework for scrutinizing geometric attributes resistant to transformational impacts. It provides a means to scrutinize and delineate characteristics that retain their essence under the sway of projective transformations, encompassing affine transformations, shifts, and scaling. These transformations act as cartographers, facilitating the mapping of points from one spatial realm to another while preserving select properties. The realm of topological transformations assumes prominence, casting light on their profound influence on geometric shapes. Topological transformations possess the power to reshape, resize, and metamorphose geometric entities, emphasizing the pivotal role of topology in comprehending the intricacies of geometry.

To further advance the field, critical avenues of research beckon. These include the pursuit of computationally efficient and stable invariants capable of accommodating broader classes of transformations, particularly projective mappings. Addressing the challenges of noisy, incomplete, and discrete data, the quest for invariants that unveil intrinsic topology amidst nonlinear coordinate transformations represents a pressing frontier. Additionally, the classification of invariants based on the structural attributes they encapsulate promises to illuminate deeper insights into the intricate interplay between geometry and transformation. This multifaceted exploration not only enriches our understanding of geometry but also holds immense potential for applications spanning mathematics, computer science, engineering, and beyond.

REFERENCES

1. Barroso-Laguna A, Brachmann E, Prisacariu VA, Brostow GJ, Turmukhambetov D. Two-view geometry scoring without correspondences. Comp Vision Pattern Recogn. 2023. https://doi.org/10.48550/arXiv.2306.01596.
2. Buzzelli M. Angle-retaining chromaticity and color space: Invariants and properties. J Imag. 2022;8(9):232. https://doi.org/10.3390/jimaging8090232.
3. Leopold C. Symmetry concepts as design approach. Aesthetics between irregularity and regularity. In: 12th SIS-Symmetry Congress “Symmetry: Art and Science” (202-209). Budapest: International Society for the Interdisciplinary Study of Symmetry, 2022. https://doi.org/10.24840/1447-607X/2022/12-25-202.
4. Bense, M. Aesthetica: Einführung in die neue Aesthetik. Agis: Verlag, 1982. Available at: https://monoskop.org/images/7/7c/Bense_Max_Aesthetica_Einfuehrung_in_die_neue_Aesthetik_1965.pdf.
5. Gözütok U, Anıl Çoban H, Sagiroglua Y. A differential approach to detecting projective equivalences and symmetries of rational 3D curves. Inf Organiz. 2023;33(1):100450. https://doi.org/10.1016/j.infoandorg.2023.100450.
6. Sergeant-Perthuis G, Rudrauf, D, Ognibene D, Tisserand Y. Action of the Euclidean versus Projective group on an agent’s internal space in curiosity driven exploration: A formal analysis. Artific Intellig. 2023. https://doi.org/10.48550/arXiv.2304.00188.
7. Wolfson HJ. Model-based object recognition by geometric hashing. In: Proceedings of the 1st European Conference on Computer Vision (526-536). Berlin: Springer, 1990. https://doi.org/10.1007/BFb0014902.
8. Gomes H. Poincaré invariance and asymptotic flatness in Shape Dynamics. Gener Relat Quant Cosmol. 2012. https://doi.org/10.48550/arXiv.1212.1755.

9. Weyl H. Symmetry. Princeton: Princeton University Press, 2017. Available at: https://www.scirp.org/(S(lz5mqp453edsnp55rrgjct55.))/reference/referencespapers.aspx?referenceid=1120383.
10. Royer FL. Detection of symmetry. Journal of Experimental Psychology: Human Percep Perform. 1981;7(6):1186-1210. https://doi.org/10.1037//0096-1523.7.6.1186.
11. Gros P, Quan L. Projective invariants for vision. 1992. Available at: https://inria.hal.science/inria-00590013/PDF/invariant.pdf.
12. Thaqi X. Learn to teach geometric transformations. Barcelona: University of Barcelona, 2009.
13. Thaqi X, Aljimi E. The structure of n harmonic points and generalization of Desargues' theorems. Mathematics. 2021;9(9):1018. https://doi.org/10.3390/math9091018.
14. Patterson BC. Projective geometry. New York: John Wiley & Sons, 1937.
15. Liberti L, Lavor C, Maculan N, Mucherino A. Euclidean distance geometry and applications. SIAM Rev. 2014;56(1):3-69. https://doi.org/10.1137/120875909.
16. Olsen J. The geometry of Möbius transformations. Rochester: University of Rochester, 2010. Available at: https://johno.dk/mathematics/moebius.pdf.
17. Flusser J, Lébl M, Šroubek F, Pedone M, Kostková J. Blur invariants for image recognition. Int J Comput Vis. 2023;131:2298-2315. https://doi.org/10.1007/s11263-023-01798-7.
18. Khadjiev D, Ören İ. Global invariants of paths and curves for the group of orthogonal transformations in the two-dimensional Euclidean space. Glob Invar Paths Curve. 2019;27(2):37-65. https://doi.org/10.2478/auom-2019-0018.
19. Nurtas M, Baishemirov Z, Aizhan Y, Aizhan A. 2-D finite element method using escript for acoustic wave propagation. In: ACM International Conference Proceeding Series (3410774). Almaty: Association for Computing Machinery, 2020. https://doi.org/10.1145/3410352.3410774
20. Kanbak C, Moosavi-Dezfooli SM, Frossard P. Geometric robustness of deep networks: analysis and improvement. 2018. Available at: https://openaccess.thecvf.com/content_cvpr_2018/papers/Kanbak_Geometric_Robustness_of_CVPR_2018_paper.pdf.
21. Berton G, Masone C, Paolicelli V, Caputo B. Viewpoint invariant dense matching for visual geolocalization. In: Proceedings of the IEEE/CVF International Conference on Computer Vision (ICCV) (12169-12178). Montreal: IEEE, 2021. https://doi.org/10.48550/arXiv.2109.09827.
22. Shynkariuk IM. Alternative representation of space and time: Geometric solution of problems of relativity theory. Sci Her Uzhhor Univ. Ser Phys. 2022;(51):74-82. https://doi.org/10.54919/2415-8038.2022.51.74-82
23. Oleksandra S, Krayushkina K, Khymerik T, Andrii B. Method of increasing the roughness of the existing road. 15th Int Sci Conf Underground Urban Prerequis Sustain Develop. 2016;165:1766-1770.
24. Bayro-Corrochano E. A survey on quaternion algebra and geometric algebra applications in engineering and computer science 1995-2020. Digit Obj Ident. 2021;9:104326-104355. https://doi.org/10.1109/ACCESS.2021.3097756.
25. Zhurakovsky YA, Korzhik VN, Borisov YS. Fine-structure of roentgen emission-spectra of alloy Ni-Nb in amorphous and crystalline states. Izv Vyssh Ucheb Zaved. Fiz. 1988;31(7):108-110.
26. Kuzhel N, Bieliatynskyi A, Prentkovskis O, Klymenko I, Mikaliunas S, Kolganova O, Kornienko S, Shutko V. Methods for numerical calculation of parameters pertaining to the microscopic following-the-leader model of traffic flow: using the fast spline transformation. Transp. 2013;28(4):413-419.

27. Sattinova ZK, Bekenov TN, Assilbekov BK, Ramazanova GI, Zhapbasbayev UK, Nussupbek ZT. Mathematical modeling of the rheological behavior of thermoplastic slurry in the molding process of beryllium ceramics. Ceram Internat. 2022. https://doi.org/10.1016/j.ceramint.2022.07.178
28. Aloev R, Berdyshev A, Akbarova A, Baishemirov Z. Development of an algorithm for calculating stable solutions of the Saint-Venant equation using an upwind implicit difference scheme. East-Eur J Enterpr Technol. 2021;4(4):47-56. https://doi.org/10.15587/1729-4061.2021.239148
29. Velich R, Kimmel R. Deep signatures – Learning invariants of planar curves. Comp Vis Pattern Recognit, 2022. https://doi.org/10.48550/arXiv.2202.05922.
30. Zhao C, Yang J, Xiong X, Zhu A, Cao Z, Li X. Rotation invariant point cloud analysis: Where local geometry meets global topology. Pattern Recognition. 2022;127:108626. https://doi.org/10.48550/arXiv.1911.00195.
31. Rudrauf D, Bennequin D, Williford K. The Moon illusion explained by the projective consciousness model. J Theor Biol. 2020;507:110455. https://doi.org/10.1016/j.jtbi.2020.110455.
32. Karpusha VD, Korzhik VN, Kunitskii YA, Poperenko LV, Zhurakovskii EA, Shaikevich IA. Electron-structure of amorphous alloy Ni60Nb40. Russ Metall. 1988;2:164-168.
33. Mustafin, A. Awakened oscillations in coupled consumer-resource pairs. J Appl Math. 2014;2014:561958. https://doi.org/10.1155/2014/561958
34. Gorda E, Serdiuk A, Nazarenko, I. Determining the invariant of inter-frame processing for constructing the image similarity metric. East-European J Ent Tech. 2023;2(122):19-25. https://doi.org/10.15587/1729-4061.2023.276650.
35. Rezaei T, Mohammadi AM, Mina E, Reza Kavianpour M. Numerical analysis of varies geometries on dynamic analysis on the gravity platform. Innovaciencia. 2019;7(2). https://doi.org/10.15649/2346075X.764
36. Chobal I, Chobal O, Myslo Yu, Petryshynets I, Rizak V. Elastic-plastic properties of Li2B4O7 determined by nanoindentation. Sci Her Uzhhor Univ. Ser Phys. 2023;(53):53-63. https://doi.org/10.54919/physics/53.2023.53
37. Aghazade P, Bagheri A, Riahi Kashani M. Designing a metaheuristic mining algorithm to separate two-color points in a two-dimensional environment. Innovaciencia. 2019;7(2). https://doi.org/10.15649/2346075X.773