APOS Theory and the Teaching of Linear Algebra: Engineering Students' Understanding of Systems of Linear Equations and Their Solution Sets

La teoría APOE y la Enseñanza del Álgebra Lineal: Comprensión de los Sistemas de Ecuaciones Lineales y su Conjunto de Solución en Estudiantes de Ingeniería

Luzdari Rangel Ruiz[1*], Solange Roa Fuentes[2], Erika García Torres[3]

Highlights

- Students' work reveals predominantly procedural reasoning when solving systems of linear equations.
- Transitions from the Action conception to the Process conception are observed, along with emerging evidence of the Object conception.
- The articulation of algebraic, matrix, and geometric representations promotes progress in understanding the solution set.

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ABSTRACT

Introduction. The teaching and learning of linear algebra remain a persistent challenge in engineering education, particularly regarding students' understanding of systems of linear equations and their associated solution sets. From the perspective of APOS Theory, these concepts are understood as mental constructions that develop through a progression from Actions to Processes and, in some cases, toward Objects, involving the articulation of algebraic, matrix, and geometric representations. **Objectives**. This study examines the mental constructions that engineering students develop when solving systems of linear equations within a teaching strategy informed by a genetic decomposition grounded in APOS Theory. **Materials and methods**. A descriptive qualitative study was conducted with 26 engineering students enrolled in a Linear Algebra course. Instruction was designed according to the ACE teaching cycle and supported by tasks intended to foster transitions among mental structures. Data sources included students' written productions and their articulated reasoning, which were analyzed using APOS Theory as an analytical framework. **Results.** The analysis reveals a progression in students' understanding of systems of linear equations and their solution sets, moving from predominantly procedural and algorithmic approaches toward reasoning that coordinates algebraic, matrix, and geometric representations. **Conclusions.** The findings suggest that a teaching strategy grounded in APOS Theory and structured through the ACE cycle supports students' development from operational treatments of systems of linear equations toward more structural forms of understanding. These results contribute to ongoing discussions on the design of instructional approaches that align with the conceptual demands of linear algebra in engineering education.

RESUMEN

Introducción. La enseñanza del álgebra lineal sigue planteando retos importantes en la formación de ingenieros, especialmente en lo que respecta a la comprensión de los sistemas de ecuaciones lineales y su conjunto de solución. Desde los postulados de la Teoría APOE, estos conceptos se conciben como construcciones mentales que evolucionan de la acción al esquema, articulando representaciones algebraicas, matriciales y geométricas. **Objetivos**. Analizar las construcciones mentales que surgen en los estudiantes de ingeniería al resolver sistemas de ecuaciones lineales, basándose en una estrategia de enseñanza diseñada con base en una descomposición genética fundamentada en la Teoría APOE. **Materiales y métodos**. Se realizó un estudio cualitativo de alcance descriptivo con 26 estudiantes de ingeniería matriculados en un curso de Álgebra Lineal. La enseñanza se organizó según el ciclo de enseñanza ACE y se apoyó en tareas que promovían la transición entre las estructuras mentales. La recolección de datos incluyó producciones escritas y razonamientos de los estudiantes, analizados a la luz de la Teoría APOE. **Resultados.** Se observó una progresión en la comprensión de los sistemas de ecuaciones lineales y su conjunto de solución, que avanzó desde la aplicación procedimental de métodos algorítmicos hacia formas de razonamiento que integran representaciones algebraicas, matriciales y geométricas. **Conclusiones.** Los hallazgos confirman que la estrategia didáctica basada en la Teoría APOE orientada por el ciclo ACE facilitó el tránsito desde concepciones operativas hacia una comprensión estructural de los sistemas de ecuaciones lineales en los estudiantes. Esto aporta criterios para diseñar propuestas de enseñanza más coherentes con la naturaleza conceptual del Álgebra Lineal en la formación de ingenieros.

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1 Universidad Autónoma de Bucaramanga, colombia *Corresponding Author: lrangel583@unab.edu.co
2 Universidad Industrial de Santander, bucaramanga, colombia, e-mail: doraroaf@uis.edu.co
3 Universidad Autónoma de Querétaro, mexico, e-mail: erika.garcia@uaq.edu.mx

INTRODUCTION

Linear algebra plays an essential role in the education of professionals across multiple disciplines, particularly in engineering, as it provides the theoretical and procedural foundations required to address a wide range of problems in diverse contexts. As noted by Stewart et al.(1) "Linear algebra is essential in Science, Technology, Engineering, and Mathematics (STEM). This role has evolved from classical engineering fields to more contemporary programs such as data science, signal pr ocessing, cryptography, and computer science" (p.183)

Enhancing students' problem-solving skills and critical thinking is valuable in any field; however, difficulties reported in various studies indicate that linear algebra can be particularly challenging for engineering students, especially regarding the understanding of systems of linear equations (SLEs) and their solution sets. These difficulties manifest as predominantly algorithmic reasoning, fragmented use of algebraic and geometric representations, and a lack of connection between procedures and their conceptual meaning. For example, students often manipulate matrices and determinants without understanding the meaning of the results obtained, nor the relationship between equations and the geometric space they represent(2–4).

Systems of linear equations constitute a point of articulation between school mathematics and university-level mathematics, where algebraic symbolization, geometric visualization, and concepts such as vector spaces, solution subspaces, and linear transformations converge(5–8). Understanding SLEs requires recognizing them not merely as a set of procedures for finding solutions, but as a mathematical object whose interpretation demands the articulation of different forms of representation: algebraic, geometric, and matrix-based(4).

In response to this issue, APOS Theory (an acronym for Action, Process, Object, Schema), proposed by Ed Dubinsky(9) provides a robust explanatory framework that makes it possible to describe the mental constructions an individual may develop when learning a mathematical concept, as well as the mechanisms through which such structures are achieved through individuals' experiences in solving particular types of mathematical tasks(10,11). In this way, it becomes possible to design cognitive models that inform instructors about the levels of understanding that an individual may attain as a result of their classroom experiences.

Based on this cognitive description, known as a Genetic Decomposition (hereafter GD), instructional tools are designed with the aim of promoting students' understanding of a particular mathematical object. However, the design and implementation of teaching strategies do not always lead to the intended learning outcomes, which invites reflection on how to assess the effectiveness of the strategies that have been designed and implemented.

From the perspective of APOS Theory, numerous studies have been conducted to analyze how certain concepts in linear algebra are learned(12,13), including matrices and systems of linear equations(2), vector spaces(14), the concept of a basis of a vector space(6,15), eigenvalues and eigenvectors(16,17) and linear transformations(18). In particular, the studies by Rodríguez Jara et al.(3) and Oliveros(4) on systems of linear equations (SLEs) and their solution sets (hereafter SS) report two GDs that describe the cognitive progression from purely algorithmic approaches to an understanding of SLEs and their solution sets as a mathematical object.

These GDs are considered a valuable tool for the design and implementation of instructional activities(19,20). However, no studies have been identified in the existing literature that use prior research findings as a foundation for designing a teaching strategy addressing specific notions such as vector spaces, linear transformations, and systems of linear equations (SLEs) along with their solution sets (SS), or even an entire linear algebra course aligned with curricular needs.

Within this context, the present study aims to analyze the mental constructions and mechanisms that emerge in engineering students when solving SLEs through a teaching strategy designed and implemented from the perspective of APOS Theory, guided by its ACE research cycle (Activities, Classroom Discussion, Exercises), as well as by the Genetic Decompositions reported by Rodríguez Jara et al.(3) and Oliveros(4).

Oktaç et al.(20) argue that there is no strict separation between teaching and research; rather, both activities share a common goal and are closely intertwined. Within the APOS theoretical framework, for example, the role of the teacher-researcher is to identify the mental structures and mechanisms necessary for students to understand a mathematical concept (whether derived from their own experience or from findings reported in prior studies), and, based on this, to design activities that foster the construction of such structures. As various studies have indicated, APOS Theory can make significant contributions to teaching and learning processes, since the cognitive models emerging from each study constitute a valuable resource for the design and implementation of instruction(20).

Trigueros and Sánchez-Matamoros(8) present a review of the teaching and learning of mathematics at the university level, highlighting persistent difficulties in understanding linear algebra, particularly with regard to SLEs. From the perspective of APOS Theory, the authors identify theoretical advances that describe the mental constructions involved in this learning process and propose incorporating the intermediate structure of Totality to refine the cognitive transition between the Process and Object structures. These findings reinforce the relevance of studying SLEs as a conceptual core of linear algebra and provide a theoretical foundation that serves as an analytical framework for the present study.

Rodríguez Jara et al.(3) analyze the cognitive construction of the solution set of a system of linear equations (SLE) with two unknowns among pre-service teachers, grounding their study in APOS Theory. Among their didactical recommendations, they highlight the need to incorporate alternative strategies for solving SLEs with three or more unknowns, to promote the articulation between geometric and algebraic representations associated with SLEs, and to design contextualized activities that foster an understanding of what constitutes a solution to a system of linear equations.

Oliveros(4) reports findings from a study conducted within the APOS theoretical framework on the understanding of SLEs. The results indicate that students tend to remain at the Action and Process levels, focusing on algebraic manipulation and the mechanical application of algorithms without establishing a relationship between the system's solution set and the solution to the problem being addressed.

The studies presented here reflect a natural cognitive development in linear algebra, in which familiarity with SLEs and their solution sets (SS) facilitates the understanding of key concepts in the field. In this regard, SLEs can be efficiently expressed using matrices and vectors. The matrix notation $Ax = b$ allows for the simplification of both analysis and manipulation, which is particularly useful in computations and practical applications. Furthermore, SLEs and their solution sets are closely linked to the notion of vector spaces. Vector spaces are sets of objects (vectors) that behave in a predictable manner under operations of addition and scalar multiplication. Understanding SLEs contributes to the comprehension of the structure and properties of vector spaces(7).

Additionally, in linear algebra, linear transformations are functions that preserve the operations of addition and scalar multiplication; SLEs and their solution sets are related to these transformations and help explain how such functions behave in terms of their effects on vectors and the vector spaces involved(18).

Regarding the solution set (SS) of an SLE, it can be expressed in different ways depending on the context and the form in which the equations are presented. It may be described in terms of variables and their values, as coordinates of points in space, or as vectors. Although notation and representation may vary, the solution set ultimately consists of all possible solutions that satisfy the constraints of the SLE. The analysis of the solution set is essential in linear algebra, as it enables an understanding of the geometry of equations and the relationships among variables. Moreover, as noted by Rodríguez Jara et al.(3), the solution set has applications in various fields, such as analytic geometry, physics, engineering, and economics, where SLEs and their solution sets are used to model and solve real-world problems.

Therefore, SLEs and their solution sets are fundamental concepts in linear algebra due to their role in algebraic formalization, geometric interpretation, and the modeling of phenomena across different domains; therefore, their understanding is essential for developing a solid mathematical foundation and for addressing complex problems in a wide range of fields.

Within this context, the present study aims to analyze the mental constructions and mechanisms that emerge in engineering students when solving systems of linear equations (SLEs) through a teaching strategy designed and implemented from the perspective of APOS Theory and guided by its ACE research cycle. To this end, APOS Theory is adopted as both a theoretical and methodological framework, as it allows for the description of the levels of understanding achieved by students. Specifically, the study addresses the following research question: How does a teaching strategy mediated by digital educational resources, oriented by APOS Theory, support engineering students' understanding of systems of linear equations and their solution sets?

APOS Theory, proposed by Dubinsky(9,21) and members of the Research in Undergraduate Mathematics Education Community (RUMEC), is a theoretical framework that analyzes how individuals come to understand mathematical concepts or notions through mental structures and mechanisms grounded in Piaget's notion of reflective abstraction. Arnon et al.(22) show that an individual makes sense of a mathematical concept when they construct certain structures, considered as states of knowledge; these include Actions, Processes, Objects, and Schemas. These structures emerge through the development of mental mechanisms such as interiorization, encapsulation, and coordination, among others, which are conceived as particular cases of reflective abstraction.

Since APOS Theory is grounded in Piaget's constructivist theory, it assumes the existence of prior Objects upon which new transformations can be performed. Thus, the construction of a mathematical concept or notion may begin with the application of an Action or a set of actions on Objects that an individual has previously constructed(18).

Actions are basic and fundamental structures for understanding a mathematical concept; they are generally the result of external prompts and are often associated with the application of algorithms. The mechanism of interiorization transforms an Action into a Process; at this level, the individual is able to reflect upon and mentally execute the Action without the need for external guidance. A Process may arise from a single Action or result from the coordination of two or more Processes. The coordination mechanism enables the individual to combine two Processes and structure them as one, allowing it to be encapsulated into an Object. Objects are static structures that allow the concept to be conceived as a whole, upon which new Actions can be applied. As shown by Arnon et al(22), the mechanism of encapsulation is complex, and few students in regular courses achieve this structure. Finally, APOS Theory introduces Schemas as the most general structures; a Schema is the collection of Actions, Processes, Objects, and other Schemas, all related to a given mathematical concept. Coherence is the main characteristic of this structure, as it allows the individual to reflect on the types of situations that can be addressed within a specific Schema.

Schemas are characterized by their continuous reconstruction and evolution, which are conditioned by the individual's mathematical activity. Their development is driven by the individual's ability to reflect on their usefulness in solving specific mathematical problems.

All the structures and mental mechanisms associated with a concept are presented in what the theory calls a genetic decomposition. These are cognitive models that describe how an individual, through the development of certain types of mental activity, may come to understand a mathematical concept at varying levels of complexity. Genetic decompositions of a concept are not unique; they are shaped by individuals' experiences, the mathematical activity generated in the classroom, and their interest in developing relationships among different mathematical concepts or notions.

APOS Theory proposes a research cycle aimed at the design and validation of genetic decompositions, which are initially hypothetical and subsequently validated through the analysis of evidence derived from individuals' work in various contexts.

This cycle consists of three components that constitute its research paradigm: Theoretical Analysis, Instructional Design and Implementation, and Observation, Analysis, and Data Verification. The theoretical analysis begins with the identification of the historical-epistemological aspects of the concept of interest, the analysis of textbooks, and the review of prior research in mathematics education related to the concept. All of this, together with the research group's experience, contributes to the development of a cognitive model for the construction of the concept, which is described in the genetic decomposition.

The theory clearly states that the genetic decomposition describes the mental structures and mechanisms, as well as the relationships among them, in order to explain how an individual achieves successfully construction of a mathematical concept(19).

The second phase is Instructional Design and Implementation, in which teaching instruments are developed based on the genetic decomposition and the ACE model is employed. This model promotes modeling, simulation, and collaborative work in the classroom. Finally, the third component, Observation, Analysis, and Data Verification, involves collecting information through didactical interviews, recordings, and transcriptions to analyze students' progress in constructing the concept, evaluating these data against the genetic decomposition as a reference(19).

MATERIALS AND METHODS

This study was conducted under a qualitative descriptive approach(23), as this framework facilitated the collection and analysis of data. The study took place at a Colombian university within the context of a linear algebra course. The group consisted of 26 engineering students enrolled in the course for the first time: 19 from the Mechatronics Engineering program and 7 from the Energy Engineering program.

In the first component, Theoretical Analysis, contributions from APOS Theory and from various studies grounded in other theoretical approaches related to the teaching and learning of linear algebra were reviewed, particularly those addressing systems of linear equations (SLEs) and their solution sets (SS). A review of the textbooks used by students was conducted, specifically *Linear Algebra* by Grossman(24) and *Linear Algebra: A Modern Introduction* by Poole(25),, and non-routine problems involving the different concepts addressed in this area were identified. From this initial analysis, the genetic decomposition (GD) proposed by Rodríguez Jara et al.(3), was adopted as a guiding framework; their work focuses on understanding the solution set of SLEs with two unknowns and proposes a genetic decomposition that progresses from homogeneous to non-homogeneous systems within a Cartesian geometric context. Additionally, the GD proposed by Oliveros(4) was considered. This combination enabled the design and implementation of a teaching strategy aimed at facilitating the transition from school mathematics to university-level mathematics, extending it to SLEs with three unknowns. Furthermore, in response to the suggestions made by Oliveros(4), efforts were made to strengthen the concept of SLEs and their solution sets by integrating them with other fundamental concepts of linear algebra, thereby avoiding the isolation of the concept from the broader context of the field.

Based on these genetic decompositions, the expected levels of understanding were defined, which allowed for a structured analysis of the cognitive development of the engineering students participating in the study and for comparison with findings reported in the literature, thus strengthening the theoretical and methodological validity of the research.

For the second component, Instructional Design and Implementation, all the tasks comprising the teaching strategy were designed and validated, and the classroom intervention was carried out. The validation process was conducted by experts in the teaching of linear algebra and APOS Theory.

The group of students received instruction grounded in APOS Theory, which emphasized active student participation from the outset and focused on fostering a deep understanding of the objects of study. Throughout the course, the use of GeoGebra software was encouraged in order to support work with multiple representations of the concepts involved. Additionally, active expression and verbal communication among all students were promoted, with continuous feedback provided and concepts contextualized through relevant engineering applications. This pedagogical approach was supported by the ACE instructional cycle, with the aim of enhancing the understanding and learning of mathematical concepts, particularly for engineering students, through active and collaborative interaction. Data collection during this phase included audio recordings, written evidence, and simulations developed by students using GeoGebra. For the third component, Observation, Analysis, and Data Verification, students' productions were analyzed based on the a priori analysis of the tasks. This made it possible to contrast the results obtained with those reported in previous research.

Teaching strategy oriented by the ACE Cycle

For the design of the teaching strategy in the linear algebra course, the institutional curriculum was taken as the starting point. This curriculum comprises seven major thematic units (Complex Numbers, Matrices and Determinants, Vectors in $\mathbb{R}^n$, Vector Spaces, Linear Transformations, Eigenvalues and Eigenvectors, and Orthogonality), which are addressed in the order presented over a 16-week period, with a total of 7 hours of instruction per week, of which 3 hours are conducted in a computer lab.

The design of the teaching strategy based on the ACE Cycle considered research findings developed within the APOS Theory framework related to the teaching of linear algebra. Systems of linear equations (SLEs) and their solution sets (SS) were identified as the central topic, given their fundamental role in the study and understanding of linear algebra and their wide range of applications in various areas of mathematics and numerous scientific and engineering fields(26).

In this study, the ACE teaching cycle was structured around the genetic decompositions (GDs) selected from the theoretical analysis(3,4). This approach enabled the organization of learning within an environment mediated by simulation and modeling, fostering students' development of a deep understanding of SLEs and their solution sets. In this regard, Franco et al.(19) argue that the GD constitutes an essential reference framework for the design and analysis of instructional activities.

As previously mentioned, the ACE teaching cycle, comprising Activities (A), Classroom Discussion (C), and Exercises (E), is a structured approach to facilitating the learning of mathematical concepts. Each component of the cycle plays a key role in promoting active and reflective learning, supporting cognitive development from direct action to deep understanding, and facilitating students' transition across the levels of Action, Process, and Object in their learning of mathematical concepts.

Within the framework of this study, the Activities component, which constitutes the first stage, includes both individual and collaborative work supported by GeoGebra software, with the aim of developing the mental structures described in the GDs(3,4) through the visualization and manipulation of different representations of SLEs and their solution sets. Therefore, in this component, and throughout the others, the focus is not solely on obtaining correct answers. The tasks that comprise this component are designed to enable students to interact with the study material by exploring and applying mathematical concepts in a concrete manner.

After completing these tasks, the Classroom Discussion phase follows, where students share their experiences and findings. This component focuses on collective reflection and analysis of the tasks performed. Classroom discussion allows students to verbalize their thoughts and reasoning, thereby facilitating the construction of a deeper and shared understanding of the material. It also provides opportunities to address doubts, correct errors, and guide discussions to deepen conceptual understanding. This component of the cycle is particularly emphasized in certain evaluative activities of the course (quizzes and in-class assignments), where students voluntarily form groups of four to discuss the tasks completed.

The final phase of the cycle, Exercises, involves students completing tasks both in class and at home that reinforce and extend their understanding of the mathematical concepts addressed during the lessons. These tasks (assigned as worksheets) are designed to consolidate learning and apply concepts to new problems or contexts. This stage enables students to practice independently and apply what they have learned, thereby strengthening their understanding. Considering the selected DG(3,4) from the Theoretical Analysis, the non-linear nature of learning, and the existence of multiple learning pathways, the teaching strategy described below was established to guide students from basic understanding to deep comprehension and application of SLEs and their solution sets.

This strategy allows for a genuine engagement with students' cognitive constructions and serves as a guide for building new knowledge based on prior structures, individualizing learning while simultaneously strengthening collaborative and experiential work. With this in mind, the strategy is presented as a progressive approach, organized into four stages, which facilitates the gradual development of students' understanding without undermining their mental constructions. APOS Theory explicitly states that all mental structures (Actions, Processes, Objects, and Schemas) are essential and do not represent hierarchical levels of learning; rather, they indicate the stage of knowledge construction without assigning quantitative or qualitative evaluations, as learning does not necessarily follow a linear progression.

In the first stage of the strategy, aimed at achieving understanding at the Action level, students begin with concrete and direct tasks. They focus on recognizing linear equations, learning to identify terms, perform basic algebraic operations, and distinguish between variables and constants. Simultaneously, they develop graphical representation skills through visualization, recalling how to draw a line, interpret slope, and identify points of intersection. The second stage, understanding at the Process level, involves tasks that promote interiorization and coordination. Here, students reflect on solution methods for equations, comparing their advantages and limitations, and develop an understanding of algebraic procedures and their geometric interpretations. At this stage, multiple equations begin to be systematized within a system, allowing students to understand how these equations interact and influence one another.

In the third stage, understanding at the Object level, processes are encapsulated into more abstract entities. Students begin to view the SLE as a cohesive whole, understanding the system beyond its individual equations and analyzing the solution set from both algebraic and geometric perspectives. At this level, students operate on SLEs and their representations without relying on specific calculations and use properties to determine the type of solution of the system, enabling them to generalize beyond particular examples.

The final stage involves the integration and application of all the concepts learned. Students apply their knowledge to complex mathematical problems and real-world contexts, developing mathematical modeling skills. Additionally, they extend their understanding to other mathematical domains, connecting their learning with advanced concepts such as matrix algebra and vector spaces. This comprehensive process allows students not only to develop an integrated understanding of SLEs and their solution sets but also to foster critical mathematical thinking skills, applying knowledge in complex and generalized contexts.

As an example, students work with simple linear equations such as $2x + 3 = 7$, performing basic operations to determine the value of $x$and deepening their algebraic and geometric interpretation of the result. The task is then extended to two variables, where students are asked to graph the line represented by the equation $2x + y = 1$ on a Cartesian plane, identifying the slope and the y-intercept. Additionally, they are asked to determine the possible values of and that satisfy the equation. Subsequently, students are given two linear equations with two unknowns and asked to graph both equations on the same plane and identify the values of and that satisfy both equations simultaneously.

The proposed task serves several important objectives. First, it strengthens basic algebraic skills (Action level). By working with linear equations in one variable, students practice fundamental algebraic operations such as addition, subtraction, and solving for variables. This part of the task helps consolidate their understanding of basic algebraic techniques and lays the foundation for the study of more complex concepts. Second, the task aims to deepen the algebraic and geometric interpretation of the obtained value (transition to the Process structure), as it enables students to begin viewing equations not merely as formulas but as representations of geometric relationships. This aspect fosters a deeper understanding of how algebraic solutions are reflected in geometric representations, which is a crucial element in linear algebra. Third, a new variable is introduced, that is, the task seeks to introduce the concept of systems of linear equations (SLEs) with two unknowns (internalization of Action and the initiation of the Process structure). The extension of the task to include two variables, along with the graphical representation of the equation $2X + 7 = 1$, introduces students to the concept of systems of linear equations. Plotting the line on a Cartesian plane helps students visualize how each equation in a system can be represented as a line and how these lines interact. By asking students to identify the slope and the y-intercept, key visualization and analytical skills are promoted.

Students begin to relate certain properties of a linear equation, such as its slope and points of intersection, through its graphical representation. These skills are essential for understanding and solving more complex systems of linear equations in the future. Students start to visualize how each equation represents a line in a Cartesian plane and to identify solutions as points that satisfy both equations.

When students are assigned two equations, the objective is to initiate the construction of the Process structure. By graphing two linear equations on the same plane and identifying points of intersection, students begin to understand the system of equations as a single mathematical object with properties that allow them to predict the type of solution. In particular, recognizing that the values of and must simultaneously satisfy both equations enables them to visualize and understand the system's solution set. In this context, the slope of the lines in a system of linear equations serves as a key indicator of the relationship between these lines and plays an important role in determining the system's solution set.

This work is subsequently extended to systems of type $2\ X\ n$, in which three or more linear equations in two variables are incorporated. This step allows students to analyze situations in which lines may intersect or be parallel, as well as cases in which some equations do not provide new information because they are linear combinations of others. This reveals relationships of dependence among the equations, strengthening students' understanding of the system's structure beyond basic geometric representation and advancing toward a more general and structural interpretation of systems of linear equations.

To promote the transition from the Process structure to the Object structure, once students identify the point of intersection in the graphical representation, or recognize its absence, they are encouraged to justify the type of solution of the system without relying exclusively on visualization or algorithmic procedures. At this stage, work with systems of size $3\ X\ 3$ is introduced in order to discuss how their geometric representation allows for the identification of the type of solution. Additionally, structural properties of systems of linear equations (SLEs), such as the determinant of the coefficient matrix, are incorporated to anticipate the type of solution the system will have and to support the selection of an appropriate solution method, particularly in cases where geometric representation is no longer viable.

To reinforce this level of abstraction and to initiate generalization beyond systems of size $2\ X\ 2$, $2\ X\ n$, and $3\ x\ 3$, tasks involving systems of other sizes are proposed, accompanied by their geometric interpretation, in order to extend the analysis to higher-dimensional cases. In these tasks, students are asked to justify the type of solution by resorting to the determinant, when it is possible to compute it, or by analyzing the relationships among equations and variables, without relying on algebraic procedures. In such cases, visual support is no longer available, which fosters abstraction and supports the understanding of SLEs as a generalizable mathematical object, linking their study to matrix algebra and vector spaces. This process prepares students to integrate SLEs and their solution sets into a broader linear algebra framework, applying them to complex problems and real-world contexts.

RESULTS

This paper presents evidence of the different structures associated with systems of linear equations (SLEs) and their solution sets (SS) that were identified among the participating students. First, evidence of collaborative work is presented to illustrate how students articulated problem-solving strategies and engaged in collective argumentation when working on tasks related to SLEs. Subsequently, individual responses collected from one of the questionnaires administered during the course are analyzed, with a specific focus on SLEs and their solution sets. Based on these productions, students' progress across each of the levels of understanding proposed by APOS Theory is presented, highlighting examples of advanced understanding, such as the use of $3 X 3$ systems and determinants. Finally, the educational impact is evaluated, reflecting on its influence on teaching practices and on the development of analytical and problem-solving skills.

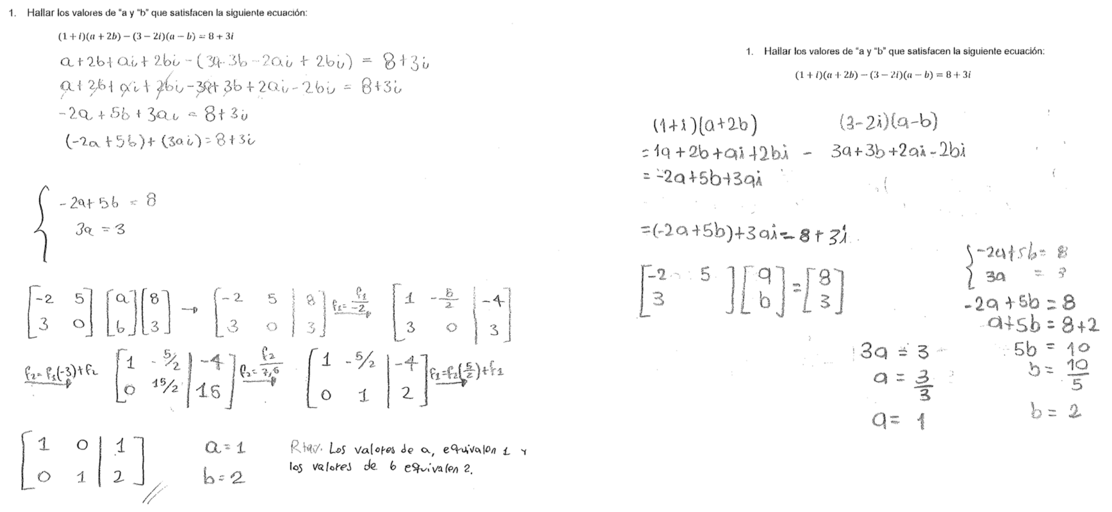
The following presents one of the tasks carried out during the Classroom Discussion phase within the ACE model, which relates SLEs and their solution sets to complex numbers. The task is adapted from Grossman[(24)].

Find the values of **a** *and* **b** *that satisfy the following equation:*

$$(1+i)(a+2b)-(3-2i)(a-bi)=8+3i$$

In this task, students were initially asked to organize themselves into groups of four. Each student was given a worksheet containing the task and allowed a maximum of 10 minutes for individual work. After this period, a discussion was initiated on how each student approached the task. During the discussion, it was observed that most students identified from the outset that the task was related to systems of linear equations, and therefore the discussion focused on the method of solution. Once the discussion concluded, the groups were dissolved, and students resumed individual work. After analyzing students' responses and procedures, it was found that all students were able to solve the task; in particular, it was observed that, following the discussion, students did not modify their initial strategies.

The procedures carried out by the two students are presented in (**Figure 1**). These not only demonstrate their ability to solve systems of linear equations with two unknowns but also reveal the understanding they have developed of underlying concepts, such as the matrix form of a system of linear equations and the Gauss-Jordan elimination method.



1. Hallar los valores de "a y "b" que satisfacen la siguiente ecuación:

$(1+i)(a+2b)-(3-2i)(a-b)=8+3i$

$a+2b+ai+2bi-(3a-3b-2ai+2bi)=8+3i$

$a+2b+ai+2bi-3a+3b+2ai-2bi=8+3i$

$-2a+5b+3ai=8+3i$

$(-2a+5b)+(3ai)=8+3i$

$$\begin{cases}-2a+5b=8\\3a=3\end{cases}$$

$$\begin{bmatrix}-2&5\\3&0\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}\begin{bmatrix}8\\3\end{bmatrix}\to\left[\begin{array}{cc|c}-2&5&8\\3&0&3\end{array}\right]\xrightarrow{f_1=\frac{f_1}{-2}}\left[\begin{array}{cc|c}1&-\frac{5}{2}&-4\\3&0&3\end{array}\right]$$

$$\xrightarrow{f_2=f_1(-3)+f_2}\left[\begin{array}{cc|c}1&-5/2&-4\\0&15/2&16\end{array}\right]\xrightarrow{f_2=\frac{f_2}{7,5}}\left[\begin{array}{cc|c}1&-5/2&-4\\0&1&2\end{array}\right]\xrightarrow{f_1=f_2(\frac{5}{2})+f_1}$$

$$\left[\begin{array}{cc|c}1&0&1\\0&1&2\end{array}\right] \quad a=1 \quad b=2$$

Rta/. Los valores de a, equivalen 1 y los valores de b equivalen 2.

1. Hallar los valores de "a y "b" que satisfacen la siguiente ecuación:

$(1+i)(a+2b)-(3-2i)(a-b)=8+3i$

$(1+i)(a+2b) \quad (3-2i)(a-b)$

$=1a+2b+ai+2bi-3a+3b+2ai-2bi$

$=-2a+5b+3ai$

$=(-2a+5b)+3ai=8+3i$

$$\begin{bmatrix}-2&5\\3&\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}8\\3\end{bmatrix}$$

$$\begin{cases}-2a+5b=8\\3a=3\end{cases}$$

$-2a+5b=8$

$a+5b=8+2$

$5b=10$

$b=\frac{10}{5}$

$b=2$

$3a=3$

$a=\frac{3}{3}$

$a=1$

Figure 1. Procedure and responses of two students.

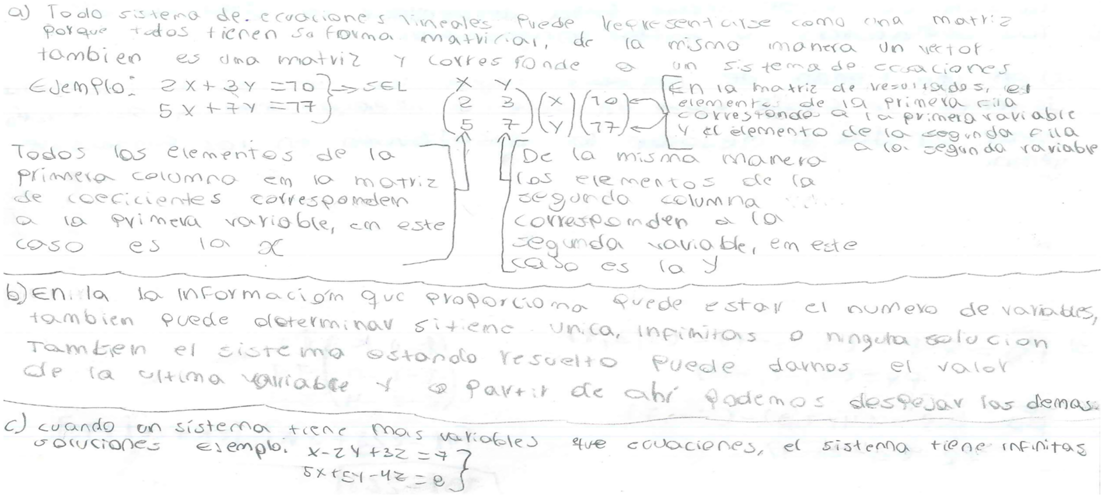
In this particular task, it was observed that students established a connection between complex numbers and systems of linear equations, allowing them to solve the task correctly. Solving this exercise demonstrated not only their ability to solve SLEs but also their command of complex numbers and their ability to articulate these concepts.

It is important to highlight that this collaborative work strategy enabled students to engage in thinking and reasoning before attempting to solve the task. Initially, it was observed that students, in most cases, responded impulsively without pausing to reflect. This aspect was addressed throughout the course, leading to increased reflection prior to engaging with mathematical tasks; this was supported by the implementation of the ACE cycle during classroom instruction.

Following the presentation of evidence from collaborative work, evidence obtained from individual work is presented below, specifically the responses collected from one of the questionnaires administered during the course, which aimed to evaluate students' understanding of SLEs and their solution sets (SS). The questionnaire consisted of three tasks, as detailed below, and was completed without the use of formulas or class notes, in order to prioritize students' reasoning and mental constructions.

Task 1. (a): How can a system of linear equations be represented using matrices and vectors? Provide an example and explain the correspondence between the elements of the matrix and the coefficients of the system.

This task made it possible to identify whether the student had constructed an Action conception, where a matrix is simply reproduced without justification, or had progressed toward a Process conception, in which the student attempts to explain the correspondence between the algebraic and matrix representations of an SLE. In the (**Figure 2**), the student's response is predominantly situated at the Action level, as the matrix representation of the system is produced based on a previously learned procedure, without resorting to structural arguments that justify such representation or to general properties of the system. The student reproduces the matrix form of an SLE as a routine action associated with symbolic manipulation.



a) Todo sistema de ecuaciones lineales puede representarse como una matriz porque todos tienen su forma matricial, de la mismo manera un vector tambien es una matriz y corresponde a un sistema de ecuaciones

Ejemplo: $\left.\begin{aligned} 2x+3y&=70 \\ 5x+7y&=77 \end{aligned}\right\} \to$ SEL

$$\begin{array}{cc} x & y \\ \end{array}$$
$$\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\begin{pmatrix} 70 \\ 77 \end{pmatrix}$$

En la matriz de resultados, el elemento de la primera fila corresponde a la primera variable y el elemento de la segunda fila a la segunda variable

Todos los elementos de la primera columna en la matriz de coeficientes corresponden a la primera variable, en este caso es la x

De la misma manera los elementos de la segunda columna corresponden a la segunda variable, en este caso es la y

b) En la la informacion que proporciona puede estar el numero de variables, tambien puede determinar si tiene unica, infinitas o ninguna solucion. Tambien el sistema estando resuelto puede darnos el valor de la ultima variable y a partir de ahí podemos despejar las demas.

c) cuando un sistema tiene mas variables que ecuaciones, el sistema tiene infinitas soluciones ejemplo: $\left.\begin{aligned} x-2y+3z&=7 \\ 5x+5y-4z&=8 \end{aligned}\right\}$

Figure 2. Student's response.

However, the explanation accompanying the representation reveals an incipient element of reflection, insofar as the student describes the correspondence between the columns of the coefficient matrix and the variables of the system. This explicit description does not yet constitute a fully developed Process conception, since no internalization of the procedure or anticipation of its effects is observed; nevertheless, it does indicate the beginning of a transition from Action to Process, as the student recognizes a functional relationship between the algebraic and matrix representations of the system. This finding is consistent with Rodríguez Jara et al.(3) , who state that the transition from an Action conception to a Process conception begins through the articulation of algebraic and matrix representations of an SLE. Similarly, Oliveros(4) argues that a student is in an intermediate phase between the Action and Process conceptions when they begin to partially explain the relationship between the algebraic and matrix representations of an SLE.

Task 2: (a) In what way does the matrix form of a system of linear equations provide information about its solution set? (b) What characteristics of a system of linear equations determine whether it has a unique solution, no solution, or infinitely many solutions? Provide examples for each case.

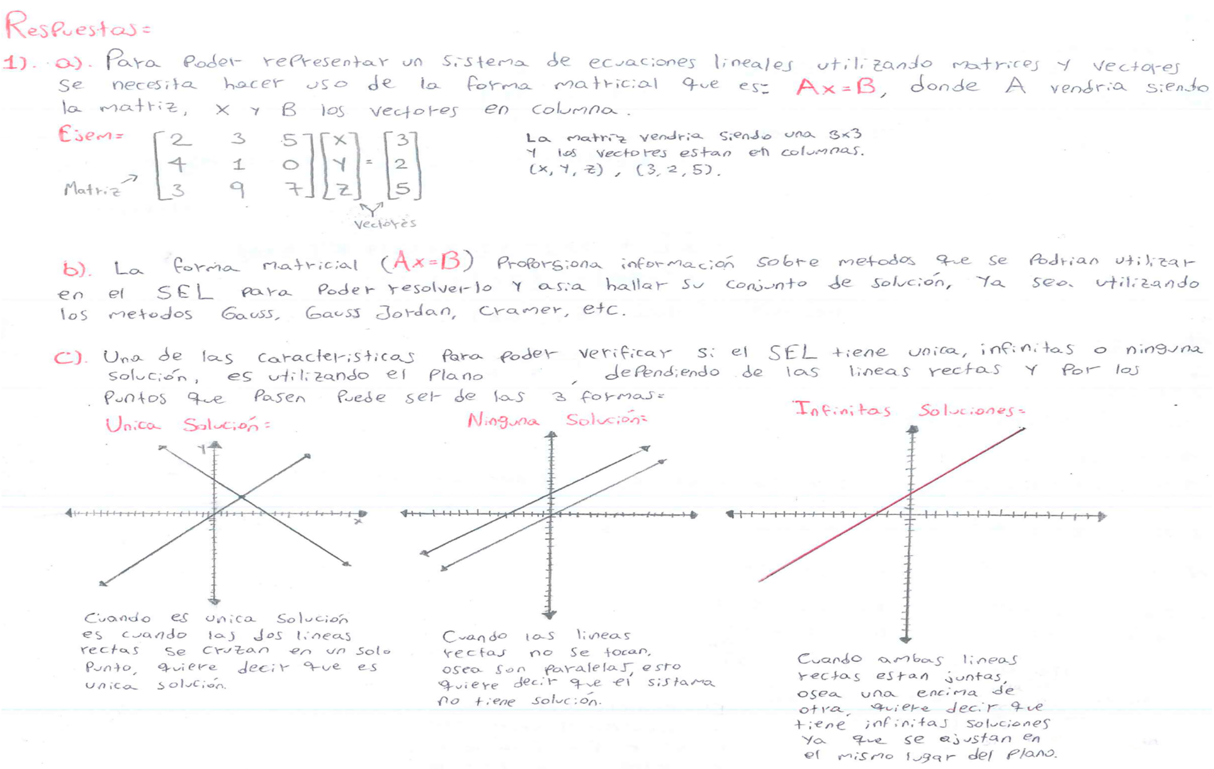
This task aimed to identify whether the student had internalized SLEs, as evidenced by their ability to interpret both algebraic and matrix representations and to infer the type of solution without explicitly solving the system. In the (**Figure 3**), the student's response shows indications of a transition toward the Process conception, although it is not yet fully consolidated.

In response to part (b), the student stated that the matrix form "provides information about the methods that could be used to solve the system." However, this reduces its function to the use of algorithmic techniques, without establishing relationships between the matrix representation and the conditions that determine the type of solution, such as the determinant of the coefficient matrix.

In response to part (c), the student reflects on the behavior of the SLE and its solution set without explicitly computing it, relating the three types of solutions to their geometric interpretation. According to Rodríguez Jara et al.(3), this suggests that the student is in an initial phase of the Process conception. Similarly, Oliveros(4) argues that students tend to first understand the geometric relationship between the system of equations and its solutions before grasping the underlying matrix structure; that is, understanding initially emerges in the geometric domain and only later, if instruction supports it, evolves toward a structural interpretation of SLEs.

From the perspective of APOS Theory, this restriction indicates that the student remains in an initial phase of the Process conception, as they operate with SLEs as particular cases learned through repetition rather than as a generalizable mathematical structure. In this sense, the fact that students' explanations are limited to the $2 X 2$ case is not merely a didactical detail, but rather reveals an epistemological barrier associated with the way linear algebra is taught; that is, students are taught to solve examples but not to conceptualize the general structure of SLEs and their relationship with higher-dimensional spaces.

Tarsk 3. Formulate a system of linear equations that has a unique solution, infinitely many solutions, and no solution.



Respuestas=

1). a). Para Poder representar un sistema de ecuaciones lineales utilizando matrices y vectores se necesita hacer uso de la forma matricial que es= Ax=B, donde A vendria siendo la matriz, x y B los vectores en columna.

Ejem=

$$\underset{\text{Matriz}}{\begin{bmatrix} 2 & 3 & 5 \\ 4 & 1 & 0 \\ 3 & 9 & 7 \end{bmatrix}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

Vectores

La matriz vendria siendo una 3x3 y los vectores estan en columnas. (x, y, z), (3, 2, 5).

b). La forma matricial (Ax=B) Proporsiona información sobre metodos que se Podrian utilizar en el SEL para Poder resolverlo y asia hallar su conjunto de solución, ya sea utilizando los metodos Gauss, Gauss Jordan, Cramer, etc.

C). Una de las caracteristicas para poder verificar si el SEL tiene unica, infinitas o ninguna solución, es utilizando el Plano, dependiendo de las lineas rectas y por los Puntos que Pasen Puede ser de las 3 formas=

Unica Solución=

Ninguna Solución=

Infinitas Soluciones=

Cuando es unica solución es cuando las dos lineas rectas se cruzan en un solo Punto, quiere decir que es unica solución.

Cuando las lineas rectas no se tocan, osea son paralelas, esto quiere decir que el sistama no tiene solución.

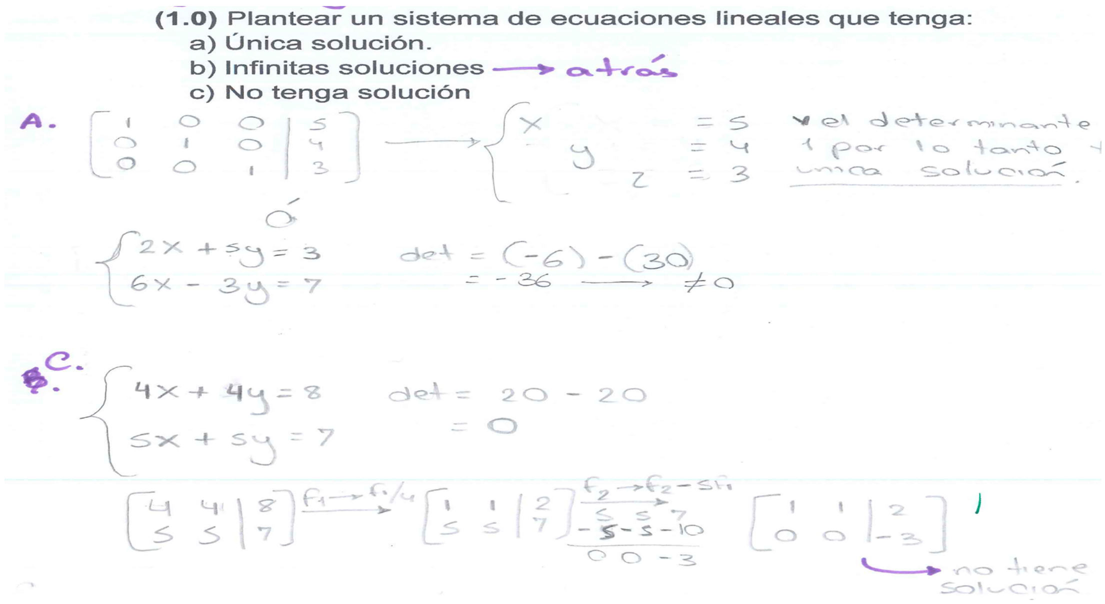
Cuando ambas lineas rectas estan juntas, osea una encima de otra, quiere decir que tiene infinitas soluciones ya que se ajustan en el mismo lugar del plano.

Figure 3. Most common response

The objective of this task is to determine whether students have constructed an Object conception of SLEs. This includes proposing systems with a unique solution, infinitely many solutions, or no solution, and justifying these constructions based on structural properties, such as relationships among rows, the determinant of the coefficient matrix, or geometric interpretation, without relying exclusively on algorithmic procedures.

In contrast to other responses in the questionnaire that remain restricted to the *2 X 2* case, in (**Figure 4**) the student formulates a *3 X 3* system and justifies the type of solution based on the value of the determinant. This is a significant finding: from the perspective of APOS Theory, it indicates a shift from a geometrically grounded understanding toward a structural one, in which the SLE is conceived independently of its graphical representation or the number of variables involved. This constitutes evidence of a transition from the Process conception to the Object conception.

In the (**Figure 5**), the response of another student is presented, who uses a system of linear equations (SLE) of size *3 X 3*. In this case, what is particularly noteworthy is the response to part (c), where the student graphically represents three parallel planes and states that "they do not intersect at a single point; therefore, there is no solution." This response is significant because the absence of a matrix-based or algebraic reference indicates that, although the student has progressed in articulating algebraic and geometric representations, they have not yet advanced toward a generalizable structural understanding. This finding confirms what has been reported by Rodríguez Jara et al.(3) and Oliveros(4), many students remain at the Process level, even when working with higher-dimensional SLEs, as long as their justification continues to rely on graphical support.



(1.0) Plantear un sistema de ecuaciones lineales que tenga:

a) Única solución.

b) Infinitas soluciones → atrás

c) No tenga solución

A. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \longrightarrow \begin{cases} x = 5 \\ y = 4 \\ z = 3 \end{cases}$ ✓ el determinante 1 por lo tanto única solución.

ó

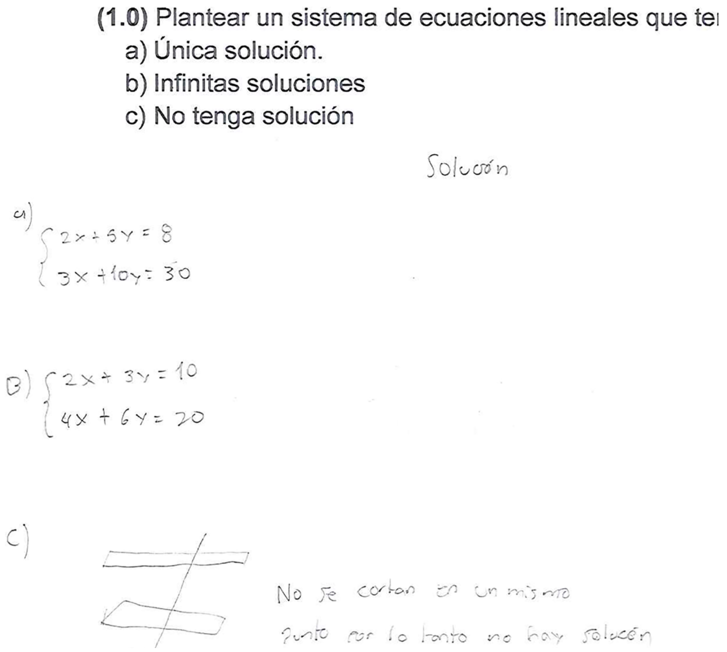
$\begin{cases} 2x + 5y = 3 \\ 6x - 3y = 7 \end{cases}$ $\det = (-6) - (30) = -36 \longrightarrow \neq 0$

C. $\begin{cases} 4x + 4y = 8 \\ 5x + 5y = 7 \end{cases}$ $\det = 20 - 20 = 0$

$\left[\begin{array}{cc|c} 4 & 4 & 8 \\ 5 & 5 & 7 \end{array}\right] \xrightarrow{f_1 \to f_1/4} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 5 & 5 & 7 \end{array}\right] \xrightarrow{f_2 \to f_2 - 5f_1} \begin{array}{ccc} 5 & 5 & 7 \\ -5 & -5 & -10 \\ \hline 0 & 0 & -3 \end{array} \quad \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & -3 \end{array}\right]$ → no tiene solución

Figure 4. Response with a SEL of size

Regarding the restriction of responses to the *2 X 2* case, this pattern was observed recurrently, despite the fact that SLEs of size *m x n* had already been addressed in class. Most examples proposed by students were limited to *2 X 2* systems, suggesting that the concept of SLEs and their solution sets has not yet been generalized beyond an elementary operational level. This preference for *2 X 2* systems may be due to their manageable complexity and the ease of geometric visualization, as each equation can be represented as a line in the plane, facilitating the understanding of concepts such as slope, intersection points, and parallel or coincident lines, and their relationship to the solution set. Additionally, *2 X 2* systems are part of the ninth-grade curriculum, which contributes to greater familiarity and confidence among students when working with them, thereby increasing their confidence when constructing arguments(27).



(1.0) Plantear un sistema de ecuaciones lineales que te

a) Única solución.

b) Infinitas soluciones

c) No tenga solución

Solución

a) $\begin{cases} 2x + 5y = 8 \\ 3x + 10y = 30 \end{cases}$

B) $\begin{cases} 2x + 3y = 10 \\ 4x + 6y = 20 \end{cases}$

c) No se cortan en un mismo punto por lo tanto no hay solución

Figure 5. **Response with SLE of size 3x3 and 2x2**

However, it is important to note that one of the common difficulties in this process is the transition from representing the solution set (SS) in two dimensions to a three-dimensional context, where there is a tendency to incorrectly assume that only parallel planes correspond to systems with no solution(27,28).

DISCUSSION

In relation to the research question, the results presented reveal differentiated progress in students' understanding of systems of linear equations (SLEs) and their solution sets. While some students remain at an operational level of understanding, based on algebraic and geometric procedures, others begin to recognize structural relationships that allow them to relate the SLE and its solution set without explicitly solving the system. These findings suggest that the design and implementation of a teaching strategy informed by APOS Theory generated didactical conditions that supported students' understanding of SLEs and their solution sets in engineering programs, while also making it possible to identify the persistence of conceptual difficulties inherent to learning linear algebra, such as the tendency to restrict reasoning to *2 X 2* systems(3,10)

From the perspective of APOS Theory, the genetic decompositions (GDs)(3,4) used in this study fulfilled a dual function: on the one hand, they allowed for the anticipation of expected mental constructions and, consequently, guided the design of the teaching strategy and the tasks implemented; on the other hand, they informed the analysis of students' work. In this sense, the GDs not only played a descriptive role in analyzing students' productions but also functioned as a didactical tool for intentionally structuring instruction(11,22)

The tasks designed for the Action level promoted the execution of elementary procedures and interaction between algebraic and geometric representations, while those oriented toward the Process level fostered articulation among representations, reflection on methods for solving SLEs, and the notion of the solution set and its coherence with the problem context(12). In this regard, some student responses provided evidence of the use of structural properties of the system, for example, the determinant of the coefficient matrix, to justify the type of solution without performing algorithmic calculations(25). From the perspective of APOS Theory, this type of reasoning is indicative of the encapsulation of the SLE as a mathematical object(22). However, these advances were not observed consistently across all students, suggesting that the consolidation of the Object conception requires more sustained and systematic didactical work, particularly oriented toward generalization to higher-dimensional SLEs.

The results of this study highlight that the proposed intervention, supported by the GDs and APOS Theory, made it possible to identify both advances and limitations in students' mental constructions regarding SLEs and their solution sets. These findings provide relevant insights for reflecting on the design of teaching strategies that intentionally address persistent difficulties in learning linear algebra.

However, the need to extend this research to larger samples and longitudinal studies is acknowledged, in order to identify, over the long term, what remains of the conceptions constructed in a course mediated by APOS Theory across different educational contexts or academic levels, as well as to further examine persistent difficulties related to generalization to higher-dimensional systems.

CONCLUSIONS

The teaching strategy designed and implemented within the framework of APOS Theory proved to be appropriate for fostering a progressive understanding of systems of linear equations (SLEs) and their solution sets (SS) among engineering students. The evidence collected allows us to assert that students evolved from the execution of routine algorithmic procedures toward forms of reasoning that articulate algebraic, matrix, and geometric representations, reflecting advances from the Action conception toward the Process conception and, incipiently, the Object conception. This evolution was manifested not only in the correct resolution of SLEs but also in the justification of the methods employed, the selection of representations, and the use of structural arguments, such as the determinant, to characterize the type of solution.

Furthermore, the results show that the ACE model and the use of tasks based on genetic decompositions (GDs) promote reflection, argumentation, and visualization, which are key elements for moving beyond mechanical problem solving and fostering a deeper understanding of linear algebra. Likewise, the incorporation of collaborative activities supported by GeoGebra contributed to strengthening students' geometric interpretation and the verbalization of mathematical ideas.

This study contributes both to the field of mathematics education in engineering and to the applied development of APOS Theory, demonstrating that its implementation not only enables the description of students' mental constructions but also guides the design of coherent teaching strategies.

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ETHICAL CONSIDERATIONS

The research received approval from the Ethics Committee of the Universidad Autónoma de Bucaramanga (UNAB) and was conducted in accordance with established ethical principles for research involving human participants. The study followed the British Educational Research Association guidelines(29) and the Belmont Report(30), ensuring respect for persons, beneficence, and justice throughout all phases of the research process. All participants were informed of the study's objectives, the voluntary nature of their involvement, and the stipulations regarding confidentiality and anonymity, in compliance with UNAB's institutional policies.

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DECLARATION OF COMPETING INTEREST

The authors declare that there is no conflict of interest in the development of this research.

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